Phase Plane Analysis

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Outline

- Why is state plane analysis useful?
- Constructing phase portrait.
- Phase plane analysis of linear systems.
- Phase plane analysis of nonlinear systems.
- Limit cycles.

Why state plane analysis?

1. Graphical interpretation.
2. Many physical systems approximately 2nd order.
3. Applicable to 1st and 2nd order systems.

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Terminology

State Plane: State space for 2nd order systems.

State Trajectory: Curve in state space.

State Portrait: plot of family of state trajectories.

Phase Plane (phase plane trajectory, phase portrait)

$$\dot{x}_1 = \alpha\, x_2,\ usually\ \alpha = 1$$

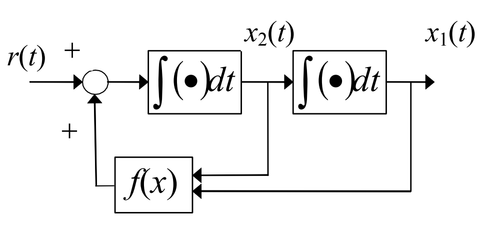
Constructing Phase Portrait.

I. Computer simulation.
II. Analytical solution: difficult in nonlinear case.
III. Graphical methods (not covered).

I. Computer Simulation

1. Obtain state-space equations.
2. Write a program to simulate the system.
3. Run simulations with different initial conditions and plot x_2 versus x_1 in each case.

Simulation Diagram



II. Analytical Solution

- LTI systems and simple nonlinear or time-varying systems.
- Two approaches:

1. Solve for state variables then eliminate *t*.
2. Eliminate *t* then solve for state variables.

Phase Plane Analysis 1st Order

- Only one state variable.
- Plot the derivative of state variable versus the state variable.

Piecewise-Linear Systems

1. Solve the equation for each linear system and plot its trajectories.
2. Join the trajectories at switching points and combine them to obtain the trajectories of the piecewise linear system.

- Trajectories and switching can be obtained by computer simulation.

Types of Equilibria: Real λ_i

- Stable node: negative eigenvalues.
- Unstable node: positive eigenvalues.
- Saddle Point: one positive and one negative eigenvalue.

Types of Equilibria: Complex λ_i

- Stable focus: LHP eigenvalues.
- Unstable focus: RHP eigenvalues.
- Center (Vortex): imaginary eigenvalues.

Phase Plane Analysis: LTI

Response characterized using the two eigenvalues of the state matrix.

$\dot{\mathbf{x}} = A\mathbf{x}$

$A \in \Re^{2\times 2}$

$\lambda_i, i = 1,2$

Real or complex conjugate.

Limit Cycle

Isolated closed curve in state space.

As $t \to \infty$

1. Trajectories converge to limit cycle (stable).
2. Trajectories diverge from limit cycle (unstable).
3. Some trajectories converge and others diverge (semistable).

Limit Cycle Stability

1. Change to polar coordinates (r, θ).
2. Find an expression for the derivative of r.

$\dot{r} = 0$ on l.c.

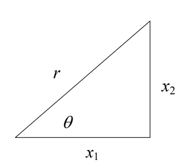
Stable: $\dot{r} > 0$ inside l.c., $\dot{r} < 0$ outside l.c.

Unstable: $\dot{r} < 0$ inside l.c., $\dot{r} > 0$ outside l.c.

Semistable: $\dot{r} > 0$ inside and outside l.c.

or $\dot{r} < 0$ inside and outside l.c.

Transformation



$r^2 = x_1^2 + x_2^2$

$\tan(\theta) = x_2 / x_1$

$\dot{r} = \dfrac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$

$\dot{\theta} = \dfrac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$