

The Z-transform

M. Sami Fadali
Professor of Electrical Engineering
UNR

1

Z-transform Definition

- **Definition 2.1** Given the causal sequence $\{u_0, u_1, u_2, \dots, u_k, \dots\}$ then its z-transform is defined as

$$\begin{aligned}U(z) &= u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_k z^{-k} + \dots \\ &= \sum_{k=0}^{\infty} u_k z^{-k} \\ z^{-1} &= \text{time delay operator}\end{aligned}$$

2

Example

Obtain the z-transform of the sequence

$$\{u(k)\} = \{1, 3, 2, 0, 4, 0, 0, 0\}$$

Solution Definition 2.1 gives

$$U(z) = 1 + 3z^{-1} + 2z^{-2} + 4z^{-4}$$

3

Z-transform Definition

- **Definition 2.2** Laplace transform the impulse train representation of sampled signal

$$\begin{aligned}u^*(t) &= u_0 \delta(t) + u_1 \delta(t-T) + u_2 \delta(t-2T) + \dots + u_k \delta(t-kT) + \dots \\ &= \sum_{k=0}^{\infty} u_k \delta(t-kT) \\ U^*(s) &= u_0 + u_1 e^{-sT} + \dots + u_k e^{-skT} + \dots \\ &= \sum_{k=0}^{\infty} u_k e^{-skT} \\ &= \sum_{k=0}^{\infty} u_k z^{-k}, \quad z = e^{sT}\end{aligned}$$

4

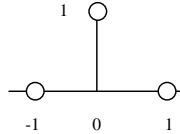
Identities Used Repeatedly

$$\sum_{k=0}^n a^k = \frac{1-a^{n+1}}{1-a}, a \neq 1$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, |a| < 1$$

5

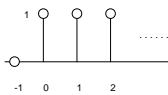
Unit Impulse

$$u(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$


- Definition 2.1: $U(z) = 1$
- Impulse-sampled version:
 $u^*(t) = \delta(t)$, Laplace transform $U^*(s) = 1$
- z-transform obtained using Definition 2.2 same as Definition 2.1

6

Sampled Unit Step



$$\{u_k\}_{k=0}^{\infty} = \{1, 1, 1, 1, 1, \dots\}$$

z-transform Definition 2.1

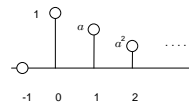
$$U(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \dots + z^{-k} + \dots$$

$$= \sum_{k=0}^{\infty} z^{-k}$$

$$U(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

7

Sampled Exponential



$$u(k) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

z-transform Definition 2.1

$$U(z) = 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots + a^k z^{-k} + \dots$$

$$= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

$$U(z) = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

8

Z-transform Properties

Linearity: Use Definition 2.2 and the linearity of the Laplace transform.

$$\mathcal{Z}\{\alpha f_1(k) + \beta f_2(k)\} = \alpha F_1(z) + \beta F_2(z)$$

Example

$$f(k) = 2 \times 1(k) + 4\delta(k), \quad k = 0, 1, 2, \dots$$

$$\begin{aligned} F(z) &= \mathcal{Z}\{2 \times 1(k) + 4\delta(k)\} \\ &= 2\mathcal{Z}\{1(k)\} + 4\mathcal{Z}\{\delta(k)\} \\ &= \frac{2z}{z-1} + 4 = \frac{6z-1}{z-1} \end{aligned}$$

9

Time Delay

Use the time delay property of the Laplace transform

$$\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$$

Example $f(k) = 4, \quad k = 2, 3, 4, \dots$

$$\begin{aligned} F(z) &= \mathcal{Z}\{4 \times 1(k-2)\} = 4z^{-2}\mathcal{Z}\{1(k)\} \\ &= z^{-2} \frac{4z}{z-1} = \frac{4}{z(z-1)} \end{aligned}$$

10

Time Advance

$$\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0)$$

$$\mathcal{Z}\{f(k+n)\} = z^n F(z) - z^n f(0) - z^{n-1} f(1) - \dots - zf(n-1)$$

Proof Apply Defn. 2.1 to $f(k+1)$

$$\begin{aligned} \mathcal{Z}\{f(k+1)\} &= \sum_{k=0}^{\infty} f(k+1)z^{-k} = z \sum_{k=0}^{\infty} f(k+1)z^{-(k+1)} \\ &= z \left\{ \left[f(0) + \sum_{k=0}^{\infty} f(k+1)z^{-(k+1)} \right] - f(0) \right\} \end{aligned}$$

Change index of summation to $m = k + 1$

$$\begin{aligned} \mathcal{Z}\{f(k+1)\} &= z \left\{ \left[\sum_{m=0}^{\infty} f(m)z^{-m} \right] - f(0) \right\} \\ &= zF(z) - zf(0) \end{aligned}$$

11

Example

Use the time advance property to find the z-transform of the causal sequence

$$\{f(k)\} = \{4, 8, 16, \dots\}$$

$$f(k) = 2^{k+2} = g(k+2)$$

$$g(k) = 2^k, \quad k = 0, 1, 2, \dots$$

$$F(z) = z^2 G(z) - z^2 g(0) - zg(1) = z^2 \frac{z}{z-2} - z^2 - 2z = \frac{4z}{z-2}$$

Easier solution:

- Write the sequence as $\{f(k)\} = 4\{1, 2, 4, \dots\} = \{4 \cdot 2^k\}$
- Use the linearity of the z-transform.

12

Discrete-Time Convolution

$$\begin{aligned}\mathcal{Z}\{f_1(k) * f_2(k)\} &= \mathcal{Z}\left\{\sum_{i=0}^k f_1(i)f_2(k-i)\right\} \\ &= F_1(z)F_2(z)\end{aligned}$$

Proof: Let the convolution give $\{y(k)\}$

$$\begin{aligned}Y(z) &= y(0) + y(1)z^{-1} + y(2)z^{-2} + \dots + y(k)z^{-k} + \dots \\ &= \sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} \left[\sum_{i=0}^k f_1(i)f_2(k-i) \right] z^{-k} \\ f(k-i) &= 0, i > k \quad (\text{causal sequence})\end{aligned}$$

13

DT Convolution Proof (cont.)

$$Y(z) = \sum_{i=0}^{\infty} f_1(i) \left[\sum_{k=i}^{\infty} f_2(k-i)z^{-k} \right]$$

$$f(k-i) = 0, i > k \quad (\text{causal sequence})$$

Change index summation index from k to $j = k-i$

$$\begin{aligned}Y(z) &= \sum_{i=0}^{\infty} f_1(i) \left[\sum_{j=0}^{\infty} f_2(j)z^{-(j+i)} \right] \\ &= \sum_{i=0}^{\infty} f_1(i)z^{-i} \left[\sum_{j=0}^{\infty} f_2(j)z^{-j} \right] = F_1(z)F_2(z)\end{aligned}$$

14

Example

Find the z-transform of the convolution of two sampled step sequences.

Solution:

By the convolution theorem, z-transform = product of the z-transforms of two step sequences.

$$\begin{aligned}F(z) &= \left(\frac{z}{z-1}\right) \times \left(\frac{z}{z-1}\right) \\ &= \left(\frac{z}{z-1}\right)^2\end{aligned}$$

15

Multiplication by Exponential

$$\mathcal{Z}\{a^{-k}f(k)\} = F(az)$$

Proof

$$\begin{aligned}LHS &= \sum_{k=0}^{\infty} a^{-k}f(k)z^{-k} \\ &= \sum_{k=0}^{\infty} f(k)(az)^{-k} \\ &= F(az)\end{aligned}$$

16

Example

Find the z-transform of the exponential sequence

$$f(k) = e^{-\alpha kT}, \quad k = 0, 1, 2, \dots$$

z-transform of a sampled step $F(z) = (1 - z^{-1})^{-1}$

$$f(k) = (e^{\alpha T})^{-k} \times 1, \quad k = 0, 1, 2, \dots$$

$$\mathcal{Z}\{(e^{\alpha T})^{-k} f(k)\} = \frac{1}{1 - (e^{\alpha T} z)^{-1}} = \frac{z}{z - e^{-\alpha T}}$$

(same as earlier example a^{-k}) 17

Complex Differentiation

$$\mathcal{Z}\{k^m f(k)\} = \left(-z \frac{d}{dz}\right)^m F(z)$$

Proof (Induction)

(i) Establish validity for $m = 1$.

(ii) Assume validity for any m and prove it for $m + 1$.

For $m = 1$, we have

$$\begin{aligned} \mathcal{Z}\{kf(k)\} &= \sum_{k=0}^{\infty} kf(k)z^{-k} \\ &= \sum_{k=0}^{\infty} f(k) \left(-z \frac{d}{dz}\right) z^{-k} = \left(-z \frac{d}{dz}\right) \sum_{k=0}^{\infty} f(k)z^{-k} = \left(-z \frac{d}{dz}\right) F(z) \end{aligned}$$

18

Proof (Cont.)

- For any m $f_m(k) = k^m f(k)$, $k = 0, 1, 2, \dots$

$$\begin{aligned} \mathcal{Z}\{kf_m(k)\} &= \sum_{k=0}^{\infty} kf_m(k)z^{-k} \\ &= \sum_{k=0}^{\infty} f_m(k) \left(-z \frac{d}{dz}\right) z^{-k} \\ &= \left(-z \frac{d}{dz}\right) \sum_{k=0}^{\infty} f_m(k)z^{-k} = \left(-z \frac{d}{dz}\right) F_m(z) \end{aligned}$$

Substitute for $F_m(z)$ $\mathcal{Z}\{kf_m(k)\} = \left(-z \frac{d}{dz}\right)^{m+1} F(z)$

19

Example

Find the z-transform of the sampled ramp sequence

$$f(k) = k, \quad k = 0, 1, 2, \dots$$

Solution: z-transform of a sampled step $F(z) = \frac{z}{z-1}$

Write $f(k)$ as: $f(k) = k \times 1$, $k = 0, 1, 2, \dots$

Apply the complex differentiation property

$$\mathcal{Z}\{k \times 1\} = \left(-z \frac{d}{dz}\right) \left(\frac{z}{z-1}\right) = (-z) \frac{(z-1) - z}{(z-1)^2} = \frac{z}{(z-1)^2}$$

20

Inversion of the z-Transform

1. **Long division:** gives as many terms of series as desired.
2. **Partial fraction expansion and table look-up:** similar to Laplace transform inversion.

21

Long Division

(i) Using long division, expand $F(z)$ as a series

$$F_i(z) = f_0 + f_1z^{-1} + \dots + f_i z^{-i}$$

$$= \sum_{k=0}^i f_k z^{-k}$$

(ii) Write the inverse transform as the sequence

$$\{f_0, f_1, \dots, f_i, \dots\}$$

22

Example

Inverse z-transform $F(z) = \frac{z+1}{z^2+0.2z+0.1}$

Solution:

(i) Long Division $z^2 + 0.2z + 0.1 \overline{) z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \dots}$

$$\begin{array}{r} z+0.2+0.1z^{-1} \\ 0.8-0.10z^{-1} \\ \underline{0.8+0.16z^{-1}+0.08z^{-2}} \\ -0.26z^{-1}-\dots \end{array}$$

$$F_i(z) = 0 + z^{-1} + 0.8z^{-2} + (-0.26)z^{-3}$$

(ii) Inverse Transformation $\{f_k\} = \{0, 1, 0.8, -0.26, \dots\}$

23

Partial Fraction Expansion

(i) Find the partial fraction expansion of $F(z)/z$.

(ii) Obtain the inverse transform $f(k)$ using the z-transform tables.

Three types of z-domain functions $F(z)$:

1. $F(z)$ with simple (non-repeated) real poles.
2. $F(z)$ with complex conjugate & real poles.
3. $F(z)$ with repeated poles.

24

I: Simple Real Roots

Residue of a complex function $F(z)$ at a simple pole z_i

$$A_i = (z - z_i)F(z) \Big|_{z \rightarrow z_i}$$

Residue = partial fraction coefficient of the i^{th} term of the expansion

$$F(z) = \sum_{i=1}^n \frac{A_i}{z - z_i}$$

25

Example

Obtain the inverse z-transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

Solution: Solve using two different methods.

(i) Partial Fraction Expansion (dividing by z)

$$\begin{aligned} \frac{F(z)}{z} &= \frac{z+1}{z(z^2 + 0.3z + 0.02)} \\ &= \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2} \end{aligned}$$

26

Example (cont.)

$$A = z \frac{F(z)}{z} \Big|_{z=0} = F(0) = \frac{1}{0.02} = 50$$

$$B = (z+0.1) \frac{F(z)}{z} \Big|_{z=-0.1} = \frac{1-0.1}{(-0.1)(0.1)} = -90$$

$$C = (z+0.2) \frac{F(z)}{z} \Big|_{z=-0.2} = \frac{1-0.2}{(-0.2)(-0.1)} = 40$$

Partial fraction expansion

$$F(z) = \frac{50z}{z} - \frac{90z}{z+0.1} + \frac{40z}{z+0.2}$$

27

Example (cont.)

(ii) Table Lookup

$$f(k) = \begin{cases} 50\delta(k) - 90(-0.1)^k + 40(-0.2)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

Note

$f(0) = 0$, so the time sequence can be rewritten as

$$f(k) = \begin{cases} -90(-0.1)^k + 40(-0.2)^k, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

28

Example (cont.)

(i) Partial Fraction Expansion (without dividing by z)

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

$$= \frac{A}{z+0.1} + \frac{B}{z+0.2}$$

Partial fraction coefficients

$$A = (z+0.1)F(z)\Big|_{z=-0.1} = \frac{1-0.1}{0.1} = 9$$

$$B = (z+0.2)F(z)\Big|_{z=-0.2} = \frac{1-0.2}{-0.1} = -8$$

29

Example (cont.)

Partial Fraction Expansion

$$F(z) = \frac{9}{z+0.1} - \frac{8}{z+0.2}$$

$$= \frac{9z}{z+0.1} z^{-1} - \frac{8z}{z+0.2} z^{-1}$$

(ii) Table Lookup (use the delay theorem)

$$f(k) = \begin{cases} 9(-0.1)^{k-1} - 8(-0.2)^{k-1}, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

(Verify: same answer as before)

30

II: Complex Conjugate & Simple Real Roots

Use the following z-transforms

$$\mathcal{Z}\{e^{-\alpha k} \sin(k\omega_d)\} = \frac{e^{-\alpha} \sin(\omega_d) z}{z^2 - 2e^{-\alpha} \cos(\omega_d) z + e^{-2\alpha}}$$

$$\mathcal{Z}\{e^{-\alpha k} \cos(k\omega_d)\} = \frac{z[z - e^{-\alpha} \cos(\omega_d)]}{z^2 - 2e^{-\alpha} \cos(\omega_d) z + e^{-2\alpha}}$$

$$z_{1,2} = e^{-\alpha} e^{\pm j\omega_d}$$

31

Residues With Complex Conjugate Poles

$$F(z) = \frac{Az}{z-p} + \frac{A^* z}{z-p^*}$$

$$f(k) = Ap^k + A^* p^{*k}$$

$$= |A| |p|^k \left[e^{j(\theta_p k + \theta_A)} + e^{-j(\theta_p k + \theta_A)} \right]$$

θ_p (θ_A) = angle of pole p (partial fraction coefficient A)

$$\text{Use: } \cos(x) = (e^{jx} + e^{-jx})/2$$

$$f(k) = 2|A||p|^k \cos(\theta_p k + \theta_A)$$

32

Example

Find the inverse z-transform of $F(z) = \frac{z^3 + 2z + 1}{(z - 0.1)(z^2 + z + 0.5)}$

Solution (i) *Partial Fraction Expansion*
Dividing by z gives

$$\frac{F(z)}{z} = \frac{z^3 + 2z + 1}{z(z - 0.1)(z^2 + z + 0.5)}$$

$$= \frac{A_1}{z} + \frac{A_2}{z - 0.1} + \frac{Az + B}{z^2 + z + 0.5}$$

$$A_1 = F(0) = -20 \quad A_2 = (z - 0.1) \frac{F(z)}{z} \cong 19.689$$

Multiply by the denominator and equate coefficients

33

Example (cont.)

Multiply by the denominator & equate coefficients

$$z^3 : A_1 + A_2 + A = 1$$

$$z^1 : 0.4 A_1 + 0.5 A_2 - 0.1 B = 2$$

A_1 and A_2 known, solve for A and B

$$A \approx 1.311 \quad B \approx -1.557$$

Check calculations

$$z^0 : -0.05 A_1 = -0.05(-20) = 1$$

$$z^2 : 0.9 A_1 + A_2 - 0.1 A + B$$

$$= 0.9(-20) + 19.689 - 0.1(1.311) - 1.557 \approx 0$$

Partial fraction expansion

$$F(z) = -20 + \frac{19.689z}{z - 0.1} + \frac{1.311z^2 - 1.557z}{z^2 + z + 0.5}$$

34

Example (cont.)

(ii) *Table Lookup* (1st two terms easy)

$$\frac{1.311z^2 - 1.557z}{z^2 - 2(-0.5)z + 0.5} = \frac{1.311z[z - e^{-\alpha} \cos(\omega_d)] - Cz e^{-\alpha} \sin(\omega_d)}{z^2 - 2e^{-\alpha} \cos(\omega_d)z + e^{-2\alpha}}$$

Equate coefficients

Denominator $e^{-\alpha} = \sqrt{0.5} = 0.707$

$$\cos(\omega_d) = -0.5 / e^{-\alpha} = -\sqrt{0.5} = -0.707$$

$$\omega_d = 3\pi/4, \text{ angle in 2nd quadrant, } \sin(\omega_d) = 0.707$$

z^1 in the numerator

$$-1.311 e^{-\alpha} \cos(\omega_d) - C e^{-\alpha} \sin(\omega_d) = -0.5(C - 1.311) = -1.557$$

$$C = 4.426$$

35

Example (cont.)

z-transform tables give

$$f(k) = -20\delta(k) + 19.689(0.1)^k$$

$$+ (0.707)^k [1.311 \cos(3\pi k/4) - 4.426 \sin(3\pi k/4)], k \geq 0$$

Trig. Identities: $\sin(A - B) = \sin(A) \cos(B) - \sin(B) \cos(A)$

$$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$$

$$4.616 = \sqrt{(1.311)^2 + (4.426)^2}$$

$$\sin^{-1}(1.311/4.616) \approx 0.288 \quad \cos^{-1}(1.311/4.616) \approx 1.283$$

$$f(k) = -20\delta(k) + 19.689(0.1)^k + 4.616(0.707)^k \sin(3\pi k/4 - 0.288)$$

$$f(k) = -20\delta(k) + 19.689(0.1)^k + 4.616(0.707)^k \cos(3\pi k/4 + 1.283)$$

36

Example (cont.)

Residue Approach

(i) Partial Fraction Expansion

Dividing by z gives

$$\frac{F(z)}{z} = \frac{z^3 + 2z + 1}{z(z-0.1)[(z+0.5)^2 + 0.5^2]}$$

$$= \frac{A_1}{z} + \frac{A_2}{z-0.1} + \frac{A_3}{z+0.5-j0.5} + \frac{A_3^*}{z+0.5+j0.5}$$

Obtain partial fraction expansion as in 1st approach

$$A_3 = \left. \frac{z^3 + 2z + 1}{z(z-0.1)(z+0.5+j0.5)} \right|_{z=-0.5-j0.5} \cong 0.656 + j2.213$$

37

Example (cont.)

$$F(z) = -20 + \frac{19.689z}{z-0.1} + \frac{0.656 + j2.213}{z+0.5-j0.5} + \frac{0.656 - j2.213}{z+0.5+j0.5}$$

Convert A_3 from Cartesian to polar form

$$A_3 = 0.656 + j2.213 = 2.308e^{j1.283}$$

Inverse z-transform to obtain

$$f(k) = -20\delta(k) + 19.689(0.1)^k + 4.616(0.707)^k \cos(3\pi k/4 + 1.283)$$

as obtained earlier ($\pi/2 - 0.288 = 1.283$).

$$f(k) = |A||p|^k [e^{j(\theta_p k + \theta_A)} + e^{-j(\theta_p k + \theta_A)}]$$

38

III: Repeated Roots

$$F(z) = \frac{N(z)}{(z-z_1)^r \prod_{j=r+1}^n (z-z_j)} = \sum_{i=1}^r \frac{A_{1i}}{(z-z_1)^{r+1-i}} + \sum_{j=r+1}^n \frac{A_j}{z-z_j}$$

$$A_{1,i} = \left. \frac{1}{(i-1)!} \frac{d^{i-1}}{dz^{i-1}} (z-z_1)^r F(z) \right]_{z \rightarrow z_1}, \quad i = 1, 2, \dots, r$$

39

Example

Obtain the inverse z-transform of the function

$$F(z) = \frac{1}{z^2(z-0.5)}$$

Solution

(i) Partial Fraction Expansion (Dividing by z)

$$\frac{F(z)}{z} = \frac{1}{z^3(z-0.5)} = \frac{A_{11}}{z^3} + \frac{A_{12}}{z^2} + \frac{A_{13}}{z} + \frac{A_4}{z-0.5}$$

40

Partial Fraction Coefficients

$$A_{11} = z^3 \frac{F(z)}{z} \Big|_{z=0} = \frac{1}{z-0.5} \Big|_{z=0} = -2$$

$$A_{12} = \frac{1}{1!} \frac{d}{dz} z^3 \frac{F(z)}{z} \Big|_{z=0} = \frac{d}{dz} \frac{1}{z-0.5} \Big|_{z=0} = \frac{-1}{(z-0.5)^2} \Big|_{z=0} = -4$$

$$A_{13} = \frac{1}{2!} \frac{d^2}{dz^2} z^3 \frac{F(z)}{z} \Big|_{z=0}$$

$$= \left(\frac{1}{2} \right) \frac{d}{dz} \frac{-1}{(z-0.5)^2} \Big|_{z=0} = \left(\frac{1}{2} \right) \frac{(-1)(-2)}{(z-0.5)^3} \Big|_{z=0} = -8$$

$$A_4 = (z-0.5) \frac{F(z)}{z} \Big|_{z=0.5} = \frac{1}{z^3} \Big|_{z=0.5} = 8$$

41

Example (cont.)

Partial Fraction Expansion

$$F(z) = \frac{1}{z^2(z-0.5)} = \frac{8z}{z^2} - 2z^{-2} - 4z^{-1} - 8$$

(ii) *Table Lookup*

z-transform tables and Definition 2.1 yield

$$f(k) = \begin{cases} 8(0.5)^k - 2\delta(k-2) - 4\delta(k-1) - 8\delta(k), & k \geq 0 \\ 0, & k < 0 \end{cases}$$

42

Example (cont.)

Evaluating $f(k)$ at $k = 0, 1, 2$, yields

$$f(0) = 8 - 8 = 0$$

$$f(1) = 8(0.5) - 4 = 0$$

$$f(2) = 8(0.5)^2 - 2 = 0$$

$$f(k) = \begin{cases} (0.5)^{k-3}, & k \geq 3 \\ 0, & k < 3 \end{cases}$$

Using the delay theorem gives the same answer.

$$F(z) = \frac{z}{z-0.5} z^{-3}$$

43

The Final Value Theorem

Theorem 2.1 The Final Value Theorem

If a sequence approaches a **constant limit** as k tends to infinity, then the limit is given by

$$f(\infty) = \lim_{k \rightarrow \infty} f(k) = \lim_{z \rightarrow 1} (1 - z^{-1})F(z)$$

$$= \lim_{z \rightarrow 1} \left(\frac{z-1}{z} \right) F(z)$$

44

Limitations of Final Value

Limit must exist for final value theorem to apply.

Does not apply to:

- (i) An unbounded sequence.
- (ii) An oscillatory sequence.

45

Proof of Final Value Thm.

Let $f(k)$ have a constant limit as k tends to infinity

$$f(k) = f(\infty) + g(k), \quad k = 0, 1, 2, \dots$$

$g(k)$ = sequence that decays to zero as $k \rightarrow \infty$

$$\lim_{k \rightarrow \infty} f(k) = f(\infty)$$

$$F(z) = \frac{f(\infty)z}{z-1} + G(z)$$

$$f(\infty) = \lim_{z \rightarrow 1} (1-z^{-1})F(z) = \lim_{z \rightarrow 1} \left(\frac{z-1}{z} \right) F(z)$$

46

Example 2.17

Verify the final value theorem using the z-transform of a decaying exponential sequence and its limit as k tends to infinity.

- **Solution:** z-transform pair

$$\{e^{-akT}\} \xleftrightarrow{\mathcal{Z}} \frac{z}{z - e^{-aT}}$$

Limit with $a > 0$ $f(\infty) = \lim_{k \rightarrow \infty} e^{-akT} = 0$

Final value theorem $f(\infty) = \lim_{z \rightarrow 1} \left(\frac{z-1}{z} \right) \left(\frac{z}{z - e^{-aT}} \right) = 0$

47

Example 2.18

Obtain the final value for the sequence whose z-transform is

$$F(z) = \frac{z^2(z-a)}{(z-1)(z-b)(z-c)}$$

What can you conclude concerning the constants a , b and c if it is known that the limit exist.

Solution: Conditions for the validity of the final value theorem $|b| < 1$ $|c| < 1$

- Apply the final value theorem

$$f(\infty) = \lim_{z \rightarrow 1} \frac{z(z-a)}{(z-b)(z-c)} = \frac{1-a}{(1-b)(1-c)}$$

48

MATLAB

$G(z)$ numerator $5(z+3)$, denominator $z^3+0.1z^2+0.4z$

num = 5*[1, 3]

den = [1, 0.1, 0.4, 0]

% Multiplication of Polynomials

denp = conv(den1, den 2)

% Partial Fraction Coefficients

[r, p, k] = residue(num, den)

p = poles, **r** = residues, **k** = coefficients of the polynomial resulting from dividing the numerator by the denominator.

49

Z-transform Solution of Difference Equations

Example 2.19: Solve the linear difference equation

$$x(k+2) - (3/2)x(k+1) + (1/2)x(k) = 1(k)$$

with the initial conditions $x(0) = 1, x(1) = 5/2$

Solution

(i) *Z-transform*

$$\begin{aligned} [z^2 X(z) - z^2 x(0) - zx(1)] - (3/2)[zX(z) - zx(0)] + (1/2)X(z) \\ = z/(z-1) \end{aligned}$$

50

(ii) Solve for $X(z)$

$$[z^2 - (3/2)z + (1/2)]X(z) = z/(z-1) + z^2 + (5/2 - 3/2)z$$

$$X(z) = \frac{z[1 + (z+1)(z-1)]}{(z-1)(z-1)(z-0.5)} = \frac{z^3}{(z-1)^2(z-0.5)}$$

51

(iii) Partial fraction expansion

The partial fraction of $X(z)/z$ is

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-0.5)} = \frac{A_{11}}{(z-1)^2} + \frac{A_{12}}{z-1} + \frac{A_3}{z-0.5}$$

$$A_3 = (z-0.5) \frac{X(z)}{z} \Big|_{z=0.5} = \frac{z^2}{(z-1)^2} \Big|_{z=0.5} = \frac{(0.5)^2}{(0.5-1)^2} = 1$$

$$A_{11} = (z-1)^2 \frac{X(z)}{z} \Big|_{z=1} = \frac{z^2}{z-0.5} \Big|_{z=1} = \frac{1}{1-0.5} = 2$$

52

Equating Coefficients

- Multiply by the denominator

$$z^2 = A_{11}(z - 0.5) + A_{12}(z - 0.5)(z - 1) + A_3(z - 1)^2$$

- Equate coefficient of z^2

$$z^2 : 1 = A_{12} + A_3 = A_{12} + 1 \quad \text{i.e. } A_{12} = 0$$

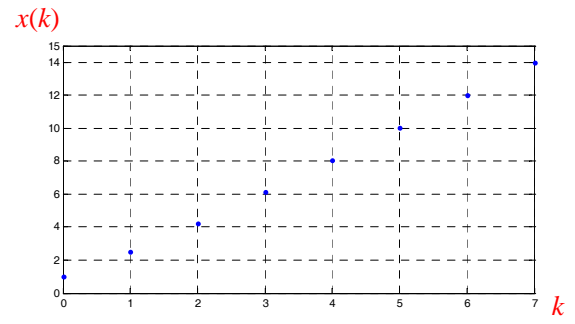
$$X(z) = \frac{2z}{(z-1)^2} + \frac{z}{z-0.5}$$

(iv) *Inverse z-transformation: z-transform tables*

$$x(k) = 2k + (0.5)^k$$

53

Plot of the Solution $x(k)$



54