Time Response of a Discrete-Time System

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Convolution Summation

Impulse response sequence

The response of a discrete time system to a unit impulse.

- Obtain response for an arbitrary input sequence in terms of impulse response.

$$\{u(k)\} = \{u(0), u(1), \ldots, u(i), \ldots\}$$

Principle of Superposition

$$u(k) = u(0)\delta(k) + u(1)\delta(k-1) + u(2)\delta(k-2) + \ldots + u(i)\delta(k-i) + \ldots$$
$$= \sum_{i=0}^{\infty} u(i)\delta(k-i)$$

Time Response

$$y(k) = u(0)h(k) + u(1)h(k-1) + u(2)h(k-2) + \ldots + u(i)h(k-i) + \ldots$$
$$= \sum_{i=0}^{\infty} u(i)h(k-i)$$

Causal System

Response starts at time i

$$h(k-i) = 0, \quad i > k$$

$$y(k) = u(0)h(k) + u(1)h(k-1) + u(2)h(k-2) + \ldots + u(k)h(0)$$
$$= \sum_{i=0}^{k} u(i)h(k-i)$$

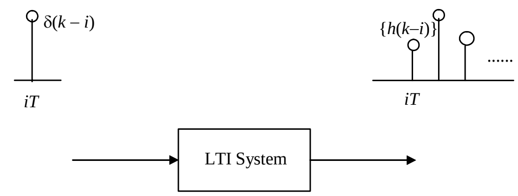
Change summation variable: $j = k - i$

$$y(k) = u(k)h(0) + u(k-1)h(1) + u(k-2)h(2) + \ldots + u(0)h(k)$$
$$= \sum_{j=0}^{k} u(k-j)h(j)$$

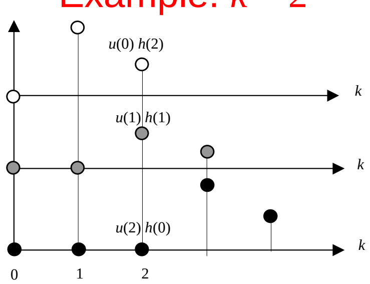
Response of a LTI System

Theorem 2.2: The response of a LTI discrete time system to an arbitrary input sequence is given by the convolution summation of the input sequence and the impulse response sequence of the system.

Response of Causal LTI DT System to an Impulse at iT



Example: $k = 2$



$$y(2) = \sum_{i=0}^{2} u(i)h(2-i)$$

$$= u(0)h(2) + u(1)h(1) + u(2)h(0)$$

Input Components & Corresponding Output Components

Input	Response
$u(0)\ \delta(k)$	$u(0)\ \{\ h(k)\ \}$
$u(1)\ \delta(k-1)$	$u(1)\ \{\ h(k-1)\ \}$
$u(2)\ \delta(k-2)$	$u(2)\ \{\ h(k-2)\ \}$

Convolution Theorem

Theorem 2.3. The z-transform of the convolution of two time sequences is equal to the product of their z - transforms.

Proof

$$Y(z) = \sum_{k=0}^{\infty} y(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \left[\sum_{i=0}^{\infty} u(i) h(k-i) \right] z^{-k}$$

Interchange summation order & let $j = k - i$

$$Y(z) = \sum_{i=0}^{\infty} \left[\sum_{j=-i}^{\infty} u(i) h(j) \right] z^{-(i+j)}$$

$$= \left[\sum_{i=0}^{\infty} u(i) z^{-i} \right] \left[\sum_{j=0}^{\infty} h(j) z^{-j} \right] = H(z) U(z)$$

Z-Transfer Function

- $H(z)$ = the *z-transfer function* or *transfer function*.
- Transfer function & impulse response sequence are z-transform pairs.
- Use the z-transform to find the system output:

i. z-transform the input.
ii. Multiply the z-transform of the input by the z-transfer function.
iii. Inverse z-transform to obtain the output sequence.

Example 2.20

$$y(k+1) - 0.5y(k) = u(k), \; y(0) = 0$$

Find the impulse response $h(k)$:
(a) From the difference equation.
(b) Using z-transformation.

(a) From Difference Equation

Let $u(k) = \delta(k)$

$$y(k+1) - 0.5y(k) = \delta(k), y(0) = 0$$

$$y(1) = 1$$

$$y(2) = 0.5y(1) = 0.5$$

$$y(3) = 0.5y(2) = (0.5)^2$$

$$\text{i.e. } h(i) = \begin{cases} (0.5)^{i-1}, & i = 1,2,3,\ldots \\ 0, & i < 1 \end{cases}$$

(b) Using z-transformation

$$y(k+1) - 0.5y(k) = u(k), y(0) = 0$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{z-0.5}$$

$$= z^{-1}\frac{z}{z-0.5}$$

Use the delay theorem

$$h(i) = \begin{cases} (0.5)^{i-1}, & i = 1,2,3,\ldots \\ 0, & i < 1 \end{cases}$$

Example 2.21

$$y(k+1) - y(k) = u(k+1), y(0) = 0$$

Find the system transfer function and its response to a sampled unit step.

Solution: z-transform

$$H(z) = \frac{z}{z-1}$$

$$Y(z) = \left(\frac{z}{z-1}\right) \times \left(\frac{z}{z-1}\right) = \left(\frac{z}{z-1}\right)^2 = z\frac{z}{(z-1)^2}$$

$$y(i) = \begin{cases} i+1, & i = 0, 1, 2, 3, \ldots \\ 0, & i < 0 \end{cases}$$