Difference Equations

M. Sami Fadali
Professor of Electrical Engineering
UNR

Outline

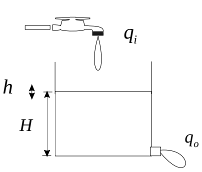
- Difference equations as models of physical systems: example.
- Types of difference equations
  - Linear.
  - Nonlinear.

DT Model for Digital Control

- Analog plant
- Piecewise constant updated periodically.
- Good approximations that allow us to obtain a simple model.
- Model obtained is called a **difference equation**.

Example: Tank Control System

- Adjust input flow rate to maintain a constant fluid level.
- Mathematically model the tank.
- Obtain a discrete-time model for the system with piecewise constant inflow q_i.



Example: Solution

- Fluid outflow and level nonlinearly related.
- Use a linear model if fluid level varies around a constant value.

h = perturbation in tank level from nominal.

q_o= perturbation in the outflow from the tank from a nominal level Q.

R = fluid resistance of the valve.

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Volumetric Balance

rate of fluid volume increase
= fluid inflow rate – fluid outflow rate

$$\frac{dCh}{dt} = (q_i + Q) - (q_o + Q)$$

- C = area of the tank = fluid capacitance.
- $\tau = R\,C$ = fluid time constant for the tank.
- Substitute from linearized valve equation

$$\frac{dh}{dt} + \frac{h}{\tau} = \frac{q_i}{C}$$

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Differential Equation Solution

$$h(t) = e^{-(t-t_0)/\tau} h(t_0) + \frac{1}{C}\int_{t_0}^{t} e^{-(t-\lambda)/\tau} q_i(\lambda) d\lambda$$

- $q_i(t) = q_i(k)$
  = constant for $t \in [k\,T, (k+1)\,T)$

- Solution over any sampling period
- Note: $h(k)$ means $h(kT)$

$$h(k+1) = e^{-T/\tau} h(k) + R\left[1 - e^{-T/\tau}\right] q_i(k)$$

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Remarks

- The discrete-time model obtained in the Example is known as a **difference equation**.
- For a linear time-invariant analog plant, we have a *linear time-invariant difference equation.*

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Difference Equations

Nonlinear Difference Equation

$$y(k+n) = f\begin{bmatrix} y(k+n-1), y(k+n-2), \ldots, y(k+1), y(k), \\ u(k+n), u(k+n-1), u(k+n-2), \ldots, u(k+1), u(k) \end{bmatrix}$$

Linear Difference Equation

$$y(k+n) + a_{n-1}y(k+n-1) + a_{n-2}y(k+n-2) + \ldots + a_1 y(k+1) + a_0 y(k)$$
$$= b_n u(k+n) + b_{n-1}u(k+n-1) + b_{n-2}u(k+n-2) + \ldots + b_1 u(k+1) + b_0 u(k)$$

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Example

Determine the order of the equation.
Is the equation (a) linear ?
(b) time-invariant ? (c) homogeneous ?

(i) $y(k+2) + 0.8y(k+1) + .07y(k) = u(k)$

(ii) $y(k+4) + \sin(0.4k)y(k+1) + .03y(k) = 0$

(iii) $y(k+1) = -0.1y^2(k)$

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Example (i)

$$y(k+2) + 0.8y(k+1) + .07y(k) = u(k)$$

- **Second order**.
- All terms linear and have constant coefficients $\Rightarrow$ **LTI.**
- A forcing function appears in the equation $\Rightarrow$ **nonhomogeneous**.

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Example (ii)

$$y(k+4) + \sin(0.4k)y(k+1) + .03y(k) = 0$$

- **Fourth order**.
- Second coefficient is time-dependent but all the terms are linear $\Rightarrow$ **linear time varying**
- No forcing function $\Rightarrow$ **homogeneous**.

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Example (iii)

$$y(k+1) = -0.1y^2(k)$$

- **First order**.
- RHS is a nonlinear function of $y(k)$ $\Rightarrow$ **nonlinear.**
- No forcing function $\Rightarrow$ **homogeneous.**
- No terms depending explicitly on time $\Rightarrow$ **time invariant.**

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