Stability

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Outline

- Stability definitions.
- Necessary conditions.
- Routh-Hurwitz criterion.
- Stable range of parameter values.
- Examples.

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Stability Definitions

- Asymptotic Stability: Natural response due to the initial conditions decays to zero exponentially as t tends to infinity.
- Unstable: unbounded natural response.
- In practice, nonlinearities limit the response as the limits of validity of the linear model are exceeded.
- Marginally Stable: response is not unbounded but does not decay exponentially.

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MATLAB Commands

```
>> roots([1,2,3,4,5])
ans =
  0.2878 + 1.4161i
  0.2878 - 1.4161i
 -1.2878 + 0.8579i
 -1.2878 - 0.8579i
>> >> zpk(g)
Zero/pole/gain:
  4 (s+1)
-------------
(s^2 + s + 4)
```

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Time Response

$$G(s)=K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_j)}=\sum_{i=1}^{n}\frac{K_i}{s-p_i}\xleftrightarrow{\mathscr{L}} g(t)=\sum_{i=1}^{n}K_i e^{p_i t},\qquad t\ge 0$$

$$G(s)=\frac{1}{(s-p_1)^2}\xleftrightarrow{\mathscr{L}} g(t)=te^{p_1 t},\qquad t\ge 0$$

$$G(s)=\frac{1}{s^2}\xleftrightarrow{\mathscr{L}} g(t)=t,\qquad t\ge 0$$

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Stability Determination

(i) *Stable:* For LHP poles [real: $p_i < 0$, complex: $\mathrm{Re}\{p_i\} < 0$], the term decays exponentially.

(ii) *Marginally stable:* If there are simple (not repeated) poles on the $j\omega$ axis, the terms do not decay but are bounded.

(iii) *Unstable:* For RHP poles or repeated $j\omega$ axis poles, the terms are unbounded.

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Closed-Loop System

- The characteristic polynomial is known but its roots (system poles) are not.

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

Necessary Stability Conditions (not enough)

coefficients $a_i > 0,\ i = 0, 1, \ldots, n$

Example: $s^2 + 0.\ s + 9$ roots at $+j3$ and $-j3$

Example: $(s+2)(s+1)(s^2+4s+10)$ stable

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First or Second Order

- Positive coefficients: necessary and sufficient stability condition.
- 1^{st} Order: LHP pole for positive coefficients.
- 2^{nd} Order: LHP poles for positive coefficients.

$$as+b=0 \qquad s=-b/a$$

$$as^2+bs+c=0 \qquad s_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Either real < b or imaginary

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Routh-Hurwitz Criterion

1) Form the Routh array.
2) All the roots of the polynomial are in the LHP **if and only if** all entries of the first column are positive.

Necessary and sufficient conditions

Routh Array

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Example

$$s^3 + 1.5s^2 + 16s + 28.5$$

Routh Array

s^3	1	16
s^2	1.5	28.5
s^1	$\dfrac{24 - 28.5}{1.5}$	
s^0	28.5	

2 sign changes: 2 RHP poles

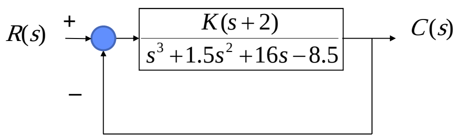
$$(s + 1.737)(s^2 - 0.2366s + 16.41)$$

Stable Range of *K*

- Use Routh-Hurwitz criterion.
- Determine the range where the first column of the Routh array remains positive.
- Multiple Constraints
  - The overall lower bound is the highest lower bound.
  - The overall upper bound is the lowest upper bound.

Example

- Find the stable range of K.



Note: The open-loop system is unstable.

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Example

$$T = \frac{K(s+2)}{s^3 + 1.5s^2 + (K+16)s + 2K - 8.5}$$

Routh Array

s^3	1	$K+16$
s^2	1.5	$2K-8.5$
s^1	$\frac{32.5-0.5K}{1.5}$	
s^0	$2K-8.5$	

$$32.5 - 0.5K > 0 \Leftrightarrow K < 65$$

$$2K - 8.5 > 0 \Leftrightarrow K > 4.25$$

$$4.25 < K < 65$$

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Zero Row

- Even polynomial

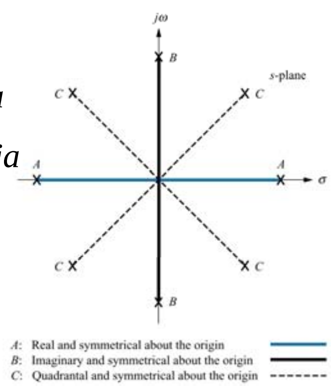
$A: s^2 - a^2 = 0 \quad s_{1,2} = \pm a$

$B: s^2 + a^2 = 0 \quad s_{1,2} = \pm ja$

$C: \left(s^2 - 2\zeta\omega_n s + \omega_n^2\right)$

$\times\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)$

$= \left(s^2 + \omega_n^2\right)^2 - \left(2\zeta\omega_n s\right)^2$



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Imaginary Axis Roots

- Auxiliary equation: use row above zero row with $K = K_{cr}$
- Solve for the roots.

s^3	1	$K+16$
s^2	1.5	$2K-8.5$
s^1	$\frac{32.5-0.5K}{1.5}$	
s^0	$2K-8.5$	

$$1.5s^2 + 2\times 65 - 8.5 = 0$$

$$s^2 = -81 \qquad s_{1,2} = \pm j9$$

$K_{cr} = 65 \quad s^3 + 1.5s^2 + 81s + 121.5 = (s+1.5)(s^2+81)$

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Multiple Constraints

- Stable range of K is the range where *all* constraints are satisfied.

Example: Routh array gives

- $K < 7$, $K > 0$, $K > -1$, $K < 10$
- Stable range $0 < K < 7$

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Quadratic Constraints

$$aK^2 + bK + c > 0$$

$$f(K) = aK^2 + bK + c = a(K - K1)(K - K2)$$

$$\frac{df(K)}{dK} = 2aK + b = 0 \text{ for max or min}$$

$$\frac{d^2 f(K)}{dK^2} = 2a$$

- for $a < 0$, f positive $K1 < K < K2$
- Minimum for $a > 0$, f positive $K < K1$, $K > K2$

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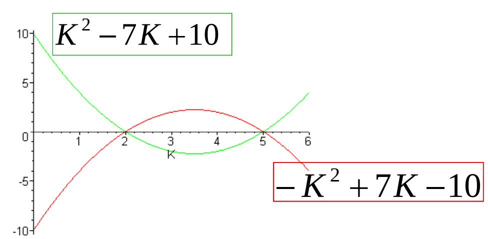
MAPLE

plot([-K^2+7*K-10, K^2-7*K+10],K=0..6)



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Example

$$G(s) = \frac{K(s^2 - s + 1)}{s^3 + 5s^2 + 13s + 77} \qquad \text{Unity feedback}$$

$$\begin{array}{c|cc} s^3 & 1 & -K + 13 \\ s^2 & K + 5 & K + 77 \\ s^1 & \dfrac{-K^2 + 7K - 12}{K + 5} & \\ s^0 & K + 77 & \end{array}$$

$$-5 < K \qquad -77 < K$$

$$-(K^2 - 7K + 12) = -(K - 3)(K - 4) > 0$$

Positive range $3 < K < 4$

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