Frequency Response

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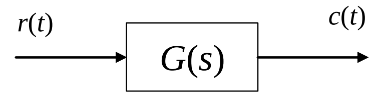
Linear System

Frequency Response $G(j\,\omega_o) = M(\omega_o)\, e^{j\phi(\omega_o)}$

Input $r(t) = A \sin(\omega_o t)$

SS Output $c(t) = A\, M(\omega_o) \sin[\omega_o t + \phi(\omega_o)]$



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Verification

$$C(s) = G(s)R(s)$$

$$= G(s)\frac{A\omega_o}{s^2+\omega_o^2}$$

$$= \frac{K_1}{s-j\omega_0} + \frac{K_1^*}{s+j\omega_0} + \sum_{i=1}^{n}\frac{B_i}{s+p_i}$$

$$K_1 = G(s)\frac{A\omega_o}{s+j\omega_0}\bigg|_{s=j\omega_o} = G(j\omega_o)\frac{A}{j2}$$

$$K_1^* = G^*(j\omega_o)\frac{A}{-j2}$$

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Steady-State Response

$$c(t) = \frac{A}{j2}\left[G(j\omega_o)e^{j\omega_o t} - G^*(j\omega_o)e^{-j\omega_o t}\right] + \sum_{i=1}^{n} B_i e^{-p_i t}$$

$$c_{ss}(t) = \frac{AM(\omega_o)}{j2}\left[e^{j[\omega_o t+\phi(\omega_o)]} - e^{-j[\omega_o t+\phi(\omega_o)]}\right]$$

$$c_{ss}(t) = AM(\omega_o)\sin[\omega_o t + \phi(\omega_o)]$$

Frequency Response: Frequency-dependent magnitude scaling and phase shift.

$G(j\,\omega_o) = M(\omega_o)\, e^{j\phi(\omega_o)}$

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Frequency Response

- Obtain the frequency response using sinusoidal inputs.
- Plot set of complex numbers vs. frequency.
- Three standard frequency response plots

1- Polar (Nyquist)

2- Bode

3- Log Magnitude vs. Phase (Nichols)

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Frequency Response Plots

1-*Polar Plot*: Each point on the plot is defined by the vector $M(\omega_i)\angle\phi(\omega_i)$, $i=1, 2, ...$

2-*Bode Plot:* 2 plots (i)20 log(M) dBs vs. ω

(ii) $\phi(\omega)°$ vs. ω

- Use $\log_{10}$ frequency axis to fit large frequency range
- One unit increase = multiplication by ten

3-*Log Magnitude vs. Phase Plot*

20 log(M) dBs vs. $\phi(\omega)°$

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Example: Simple Lag

$$G(s)=\frac{K}{s/\omega_b+1}\xrightarrow{s=j\omega}G(j\omega)=\frac{K}{j\omega/\omega_b+1}$$

$$G(j\omega)=\frac{K}{\sqrt{(\omega/\omega_b)^2+1}}\angle-\tan^{-1}(\omega/\omega_b)$$

$$M(\omega)=\frac{K}{\sqrt{(\omega/\omega_b)^2+1}},\qquad \phi(\omega)=-\tan^{-1}(\omega/\omega_b)$$

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3 dB Point

$$M(\omega_b)=\frac{K}{\sqrt{2}}\qquad \phi(\omega_b)=-45^\circ$$

$$M(\omega_b)\text{dB}=20\log\left(\frac{K}{\sqrt{2}}\right)$$

$$=20\log(K)-20\times\frac{1}{2}\times\log(2)$$

$$\approx 20\log(K)-3$$

ω_b is the 3 dB frequency (point)

break frequency or *corner frequency*

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Asymptotic Behavior

- Examine the frequency response
1. In the limit as $\omega \to 0$
2. In the limit as $\omega \to \infty$
- Use the information to **sketch** the frequency response.

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Asymptotic Behavior: Simple Lag

a) $$G(s) = \frac{K}{s/\omega_b + 1} \xrightarrow{s=j\omega} G(j\omega) = \frac{K}{j\omega/\omega_b + 1}$$

Small $|s|$: $G(s) \approx K$

$$\mathop{Lim}_{s \to j0^+} G(s) = K\angle 0^\circ \text{ or } 20\log(K)\,\text{dBs}\angle 0^\circ$$

Large $|s|$ $\quad G(s) \approx \dfrac{K}{s/\omega_b}$

$$\mathop{Lim}_{s \to j\infty} G(s) = 0\angle -90^\circ \text{ or } -\infty\,\text{dBs}\angle -90^\circ$$

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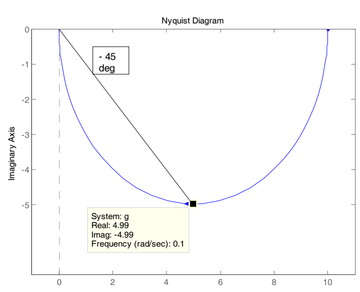
Polar Plot: Simple Lag



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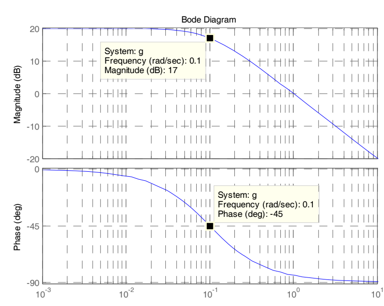
Bode Plot: Simple Lag



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Asymptotic Behavior: Type 0, *n*th order (no zeros)

$$G(s)=\frac{K}{a_n s^n + a_{n-1}s^{n-1}+\cdots+a_1 s+1}$$

$$\text{Small}|s|: G(s)\approx K$$

$$\mathop{\mathcal{L}im}_{s\to j0^+} G(s)=K\angle 0^\circ \text{ or } 20\log(K)\,\text{dBs}\angle 0^\circ$$

$$\text{Large}|s|: G(s)\approx \frac{K}{a_n s^n}$$

$$\mathop{\mathcal{L}im}_{s\to j\infty} G(s)=0\angle -90^\circ\times n \text{ or } -\infty\,\text{dBs}\angle -90^\circ\times n$$

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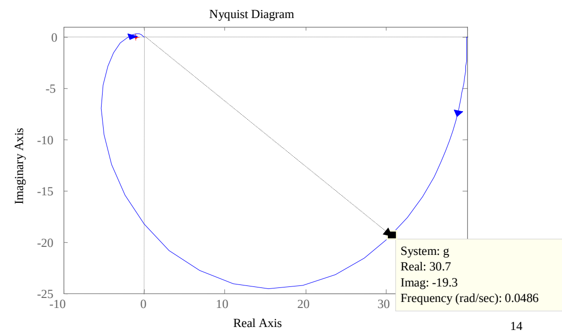
Polar Plot: 3rd order, type 0 system (no zeros)



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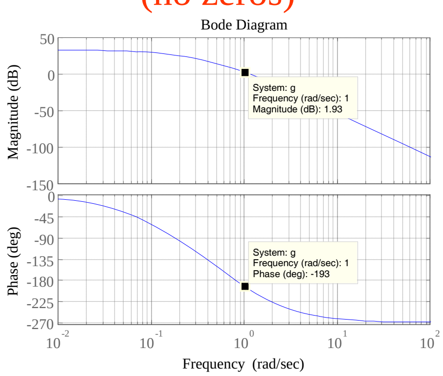
Bode Plot: 3rd order, type 0 system (no zeros)



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Asymptotic Behavior: Type 0, *n*th order, *m* zeros

$$G(s)=K\frac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_1 s+1}{a_n s^n + a_{n-1}s^{n-1}+\cdots+a_1 s+1}$$

$$\text{Small}|s|: G(s)\approx K,\ \mathop{\mathcal{L}im}_{s\to j0^+} G(s)=K\angle 0^\circ \text{ or } 20\log(K)\,\text{dBs}\angle 0^\circ$$

$$\text{Large}|s|: G(s)\approx \frac{Kb_m s^m}{a_n s^n}$$

$$\mathop{\mathcal{L}im}_{s\to j\infty} G(s)=\begin{cases}0\angle -90^\circ(n-m), & n>m\\ K\,b_n/a_n\angle 0^\circ, & n=m\end{cases}$$

$$\text{or}\begin{cases}-\infty\text{dBs}\angle -90^\circ(n-m), & n>m\\ 20\log(K\,b_n/a_n)\text{dBs}\angle 0^\circ, & n=m\end{cases}$$

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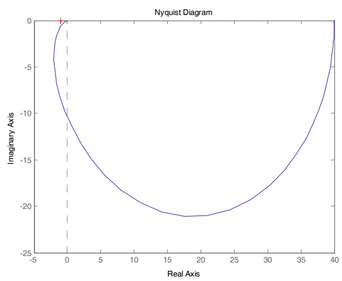
Type 0, 3rd order, 1 zero

>> s=tf('s'); g = 40*(s/0.4+1)/(s/0.1+1)/(s/0.5+1) /(s+1)



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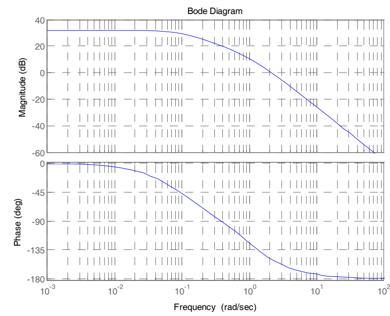
Type 0, 3rd order, 1 zero

>> s=tf('s'); g = 40*(s/0.4+1) /(s/0.1+1)/(s/0.5+1) /(s+1)



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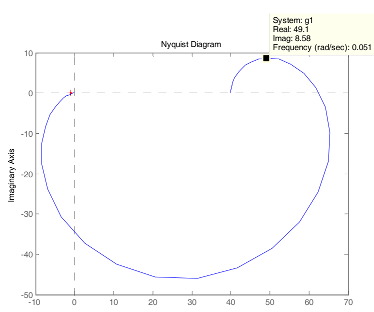
Type 0, 3rd order, 1 zero

>> s=tf('s'); g = 40*(s/0.05+1) /(s/0.1+1)/(s/0.5+1) /(s+1)



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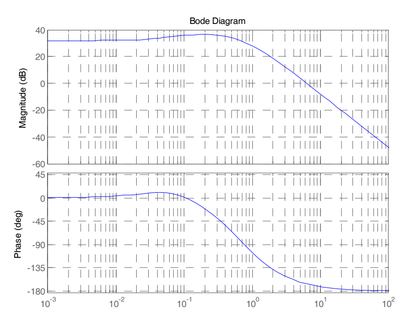
Type 0, 3rd order, 1 zero

>> s=tf('s'); g = 40*(s/0.05+1) /(s/0.1+1)/(s/0.5+1) /(s+1)



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Asymptotic Behavior: Type l, nth order (no zeros)

$$G(s) = \frac{K}{s^l\left(a_{n-l}s^{n-l} + a_{n-l-1}s^{n-l-1} + \cdots + a_1 s + 1\right)}$$

Small $|s|$ $\quad G(s) \approx \dfrac{K}{s^l}$

$$\mathop{\mathcal{Lim}}_{s \to j0^+} G(s) = \infty\angle -90l^\circ \text{ or } \infty \text{dBs}\angle -90l^\circ$$

Large $|s|$ $\quad G(s) \approx \dfrac{K}{a_{n-l}s^n}$

$$\mathop{\mathcal{Lim}}_{s \to j\infty} G(s) = 0\angle -90n^\circ \text{ or } -\infty \text{dBs}\angle -90n^\circ$$

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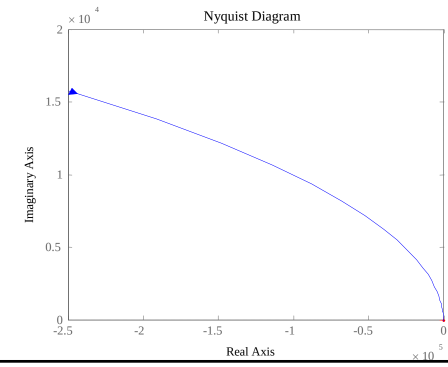
Polar Plot: 3rd order, type II system (no zeros)



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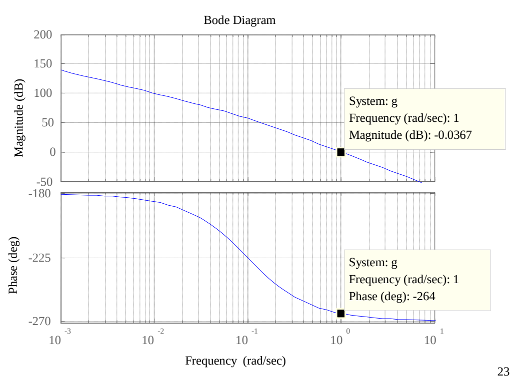
Bode Plot: 3rd order, type II system (no zeros)



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Asymptotic Bode Plots

$$G(s) = \frac{1}{s} \xrightarrow{s=j\omega} G(j\omega) = \frac{1}{j\omega}$$

Slope = –20 dB/decade = –6 dB/octave

–20 log(ω) vs. ω on a log scale

1 unit increase in log(ω) $\Rightarrow 10\omega$ (decade)

–20 log(10ω)= –20 log(ω) – 20

–20 log(2ω) = –20 log(ω) – 6

For n integrators, multiply slope & angle by n.

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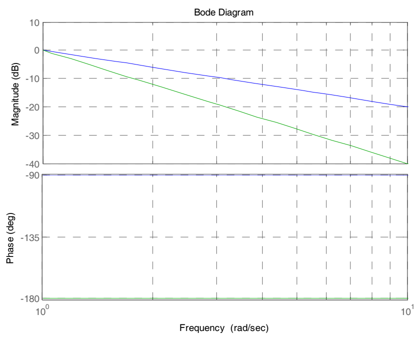
Bode Plots of Integrators



Simple Lag

$$G(s)=\frac{1}{s/\omega_b+1}\xrightarrow{s=j\omega}G(j\omega)=\frac{1}{j\omega/\omega_b+1}$$

Small $|s|$ $\quad G(s)\approx 1$ $\qquad$ Large $|s|$ $\quad G(s)\approx 1/(s/\ \omega_b)$

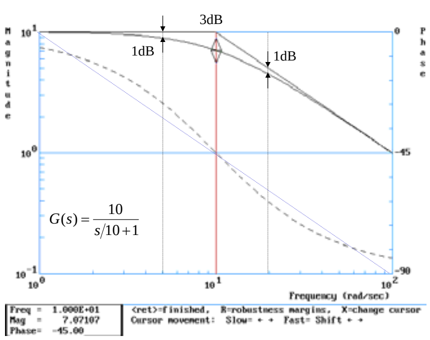
Asymptotes

Low frequency: M = 0 dB (horizontal line)

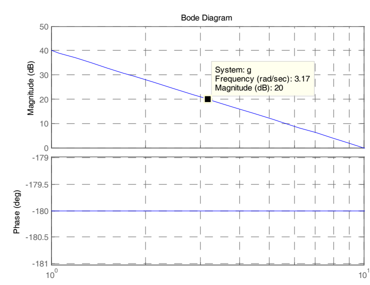
High frequency: line with

- slope = –20 dB/decade = –6 dB/octave
- 0 dB intercept (M = 1) at $\omega = \omega_b$

Bode Plot & Asymptotes for Simple Lag



Intercept

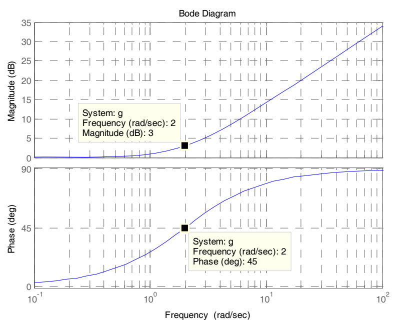


$$G(s)=K/s^n \quad |G(j\omega)|=K/\omega^n$$

$$K/\omega_{0dB}^n=1 \qquad \omega_{0dB}=\sqrt[n]{K}\ rad/s$$

$$\omega=1rad/s:|G(j1)|=K$$

Bode Plot of Zero



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2nd Order Underdamped System

$$G(s) = \frac{1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

$$\xrightarrow{s = j\omega} G(j\omega) = \frac{1}{1 - (\omega/\omega_n)^2 + j2\zeta(\omega/\omega_n)}$$

$$M(\omega) = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + 4\zeta^2(\omega/\omega_n)^2}}$$

$$\phi(\omega) = -\tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right]$$

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Peak Magnitude & Frequency

$$M^{-2}(x) = [1 - x]^2 + 4\zeta^2 x$$

$$x = (\omega/\omega_n)^2$$

$$\frac{dM^{-2}(x)}{dx} = 2[1 - x] + 4\zeta^2 = 0$$

$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}$$

$$M_p = \frac{1}{2\zeta\sqrt{1 - 2\zeta^2}}$$

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Effect of Changing ζ

- Peak amplitude & location change with ζ.
- Increasing the damping ratio gives:

1. A smaller peak M_p.
2. A lower peak frequency ω_p.
3. A slower rate of change in phase.
4. Peak disappears at

$$\zeta \geq 1/\sqrt{2} \approx 0.7$$

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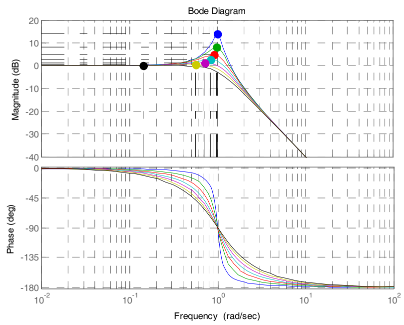
Bode Plot of 2^{nd} Order Underdamped System ζ=0.1-0.7



Sketching Asymptotic Bode Plots Type *l* , *n poles, m* zeros

Magnitude

1. Type *l*: Low frequency slope = –20 *l* dB/dec = –6 *l* dB/oct with intercept 20 log *K* at ω=1 rad/s or 0 dB at $(K)^{1/l}$.
2. Change slope at each break frequency (+20 db/dec for a zero, –20 db/dec for a pole).

Phase:

1. Approach –90 *l* º at low frequencies & –90(*n*–*m*) º as high frequencies.
2. Use calculator for a few additional points.