

Wiener Filtering Using the Invert/Transform Integral

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Abstract

The invert transform approach can be useful in teaching subjects where convolutions are introduced or used, as in signals and systems courses. In this paper, we demonstrate that the approach can also simplify the presentation and implementation of the Wiener Filter when introduced to advanced undergraduate or entry level graduate students in statistical signal processing and Kalman filtering courses.

I. Introduction

In the 1940s Wiener published his ideas for what later came to be known as Wiener filtering [1]. The Wiener filter yields the optimal estimate in the least squares sense of a random signal, based on observations that are a linear function of the signal, corrupted with additive random noise. The optimal filter can be derived either using a variational approach or the orthogonality principle. Either approach yields an integral equation to be solved for the filter weighting function, the impulse response of a causal linear system [2], [3], [4]. The integral equation in the stationary noncausal case is easily solved by two-sided or bilateral Laplace transform to yield the Wiener smoother. The causal filtering problem, however, requires the solution of the Wiener-Hopf equation, a Fredholm integral equation of the first kind, whose solution is quite complex, especially in the nonstationary case. We provide an alternative approach to the solution of the integral equation of noncausal Wiener filtering using an invert/transform integral [5]. This formula is useful in any situation making use of convolution, such as signals and systems courses. For brevity, we only examine the case of a filter with distinct poles. However, the approach can be extended to the case of repeated poles using the derivatives of the Dirac delta function [5]. The new approach was presented to a sample of students already familiar with the Wiener filter. Their responses indicate that the approach simplifies the presentation of the Wiener filter to advanced undergraduate and beginning graduate students in statistical signal processing and Kalman filtering courses.

The paper is organized as follows. In Section II, we present the invert/transform result on which we base the paper. We apply the approach to the convolution integral in Section III. In Section IV, we summarize the Wiener filtering problem and present its solution. We apply the invert/transform result to the stationary noncausal Wiener filter in Section V, and to the nonstationary noncausal Wiener filter in Section VI. In each of these sections, we provide a numerical example. Section VII provides a discussion of the results of the paper and conclusions.

II. Invert/Transform

In this section we present a useful expression derived in [5] for the integral of the product of two functions in terms of the Laplace transform of the first and the inverse transform of the second. Let $f(t)$ be a real-valued function of exponential order on \mathcal{R}^n and recall the definitions of the Laplace transform and its inverse [6]

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st} dt \\ f(t) &= \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds \end{aligned} \quad (1)$$

where s is a complex variable and σ is selected to guarantee convergence. The requirement that $f(t)$ be of exponential order guarantees the existence of the Laplace transform.

The following lemma was derived in [5].

Lemma 1: If $f_i(t)$, $i = 1, 2$, are real-valued functions of exponential order on \mathcal{R}^n , then the integral of the product

$$\begin{aligned} \int_{-\infty}^{\infty} f_1(t)f_2(t)dt &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\{f_1(t)\}\mathcal{L}\{f_2(t)\}dt \\ &= \int_{-\infty}^{\infty} \mathcal{L}\{f_1(t)\}\mathcal{L}^{-1}\{f_2(t)\}dt \end{aligned} \quad (2)$$

Proof

Using the Laplace transform of the product of two time functions [6], we have

$$\begin{aligned} \int_{-\infty}^{\infty} f_1(t)f_2(t)dt &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\left\{\mathcal{L}[f_1(t)f_2(t)]\right\}dt \\ &= \int_{-\infty}^{\infty} \mathcal{L}^{-1}\left\{\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F_1(u)F_2(s-u)du\right\}dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi j} \left\{ \int_{\sigma-j\infty}^{\sigma+j\infty} e^{st} \left[\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F_1(u)F_2(s-u)du \right] ds \right\} dt \end{aligned}$$

If all integrals are convergent, we can interchange the order of integration. Thus, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_1(t)f_2(t)dt &= \int_{-\infty}^{\infty} F_1(u) \left\{ \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} e^{ut} \left[\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F_2(s-u)e^{(s-u)t} ds \right] dt \right\} du \\
&= \int_{-\infty}^{\infty} F_1(u) \left\{ \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} e^{ut} f_2(t) dt \right\} du \\
&= \int_{-\infty}^{\infty} \mathcal{L}\{f_1(t)\} \mathcal{L}^{-1}\{f_2(t)\} dt
\end{aligned}$$

The proof of the second form of the integral follows by interchanging $f_1(t)$ and $f_2(t)$. •

Although we derived (2) for the Laplace transform, the expression can be used with other linear transforms. The expression is particularly useful with Laplace transforms because it converts exponential into Dirac delta functions. The sifting property of the delta function simplifies the evaluation of the integral considerably. However, if the expression is used with causal functions, step functions must be used in the integrand and the limits of the integrals must be carefully considered. These ideas are illustrated in the following section.

III. The Convolution Integral

The convolution integral plays an important role in linear system theory. The response $y(t)$ of any linear time-invariant system to an input $u(t)$ can be expressed in terms of the impulse response $h(t)$ as the convolution

$$y(t) = \int_{-\infty}^{\infty} h(\xi)u(t-\xi)d\xi = \int_{-\infty}^{\infty} h(t-\xi)u(\xi)d\xi \quad (3)$$

We show how the convolution integral for any input can be simplified using the invert/transform approach. For a n th order causal linear system with distinct poles, we have [6]

$$h(t) = \sum_{i=1}^n z_i e^{\lambda_i t} 1(t) \quad (4)$$

where, adopting Ogata's notation [6], $1(t)$ denotes a unit step function, or

$$1(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad (5)$$

Applying (2), we rewrite (3) as

$$y(t) = \sum_{i=1}^n z_i e^{\lambda_i t} \int_{-\infty}^{\infty} \delta(\xi - \lambda_i) \mathcal{L}\{u(\xi)1(t - \xi)\} d\xi \quad (6)$$

Note that the integral is now quite simple because the exponential terms are replaced by Dirac delta functions. The sifting property of the delta function yields [6]

$$\begin{aligned} y(t) &= \sum_{i=1}^n z_i e^{\lambda_i t} \left[\mathcal{L}\{u(\xi)1(t - \xi)\} \right]_{s=\lambda_i} \\ &= \sum_{i=1}^n z_i e^{\lambda_i t} \left[\mathcal{L}\{u(\xi)[1 - 1(\xi - t)]\} \right]_{s=\lambda_i} \end{aligned} \quad (7)$$

Example 1

Obtain the response of the system

$$h(t) = [z_1 e^{-2t} + z_2 e^{-4t}]1(t)$$

with $z_i, i=1, 2$, constant, to the causal input $u(t) = e^{-3t}1(t)$.

Using (7) with positive final time t_f gives the response

$$\begin{aligned} y(t_f) &= \sum_{i=1}^2 z_i e^{\lambda_i t_f} \left[\frac{1 - e^{-3t_f} e^{-st_f}}{s + 3} \right]_{s=\lambda_i} \\ &= z_1 e^{-2t_f} \frac{1 - e^{-3t_f} e^{2t_f}}{-2 + 3} + z_2 e^{-4t_f} \frac{1 - e^{-3t_f} e^{4t_f}}{-4 + 3} \end{aligned}$$

Simplifying gives the output

$$y(t_f) = z_1 e^{-2t_f} - z_2 e^{-4t_f} + (z_2 - z_1) e^{-3t_f}, t_f \geq 0$$

An identical expression can be obtained using the Laplace transform of the response

$$Y(s) = H(s)U(s) = \frac{z_1}{(s + 2)(s + 3)} + \frac{z_2}{(s + 4)(s + 3)}$$

The advantage of our approach is that it avoids direct evaluation of the convolution in the time domain but does not require Laplace transformation and partial fraction expansion in the s-domain.

IV. *A Review of the Wiener Filter*

Let $y(t)$ be the signal of interest, $n(t)$ be additive noise, and the available measurement be the noise corrupted signal equal to their sum of $y(t) + n(t)$. To estimate the signal $y(t)$ we use a linear filter with impulse response $h(t, \tau)$ to filter the measurement. The estimate is thus the output of the linear filter

$$\hat{y}(t + \alpha) = \int_{t_{\min}}^{t_{\max}} h(t, \tau)[y(\tau) + n(\tau)]d\tau, \quad t_{\min} \leq \tau \leq t_{\max} \quad (8)$$

with α a design parameter. For positive α we have a prediction problem, for negative α we have a smoothing problem, and for zero α we have a filtering problem. The Wiener filter seeks the linear estimate $\hat{y}(t + \alpha)$ that minimizes the expected value of the squared error. Specifically, we seek

$$E\{[y(t + \alpha) - \hat{y}(t + \alpha)]^2\} \quad (9)$$

where $E(\cdot)$ is the expectation operator. For a time-invariant filter, the impulse response reduces to the simpler form $h(t - \tau)$.

It can be shown that the optimal filter satisfies the equation [3], [4]

$$\int_{t_{\min}}^{t_{\max}} h(\tau, u)R_{y+n}(u)du = R_{y+n, y}(\alpha + \tau), \quad t_{\min} \leq \tau \leq t_{\max} \quad (10)$$

where $R_{y+n}(t)$ is the autocorrelation of the measurement $y(t) + n(t)$ and $R_{y+n, y}(t)$ is the cross-correlation of the measurement and the signal of interest.

The integral equation (10) is quite difficult to solve in its general form. However, particular special cases of the equation are much easier. For example, the original work of Wiener considered the steady-state problem with both t_{\min} and t_{\max} infinite. The resulting stationary noncausal problem is easily solved using two-sided Laplace transformation [3]. We are more interested in the causal stationary and nonstationary problems, which we can simplify using the transform/invert formula [5].

V. *Causal Wiener Filter: The Stationary Case*

For a causal filter, it can be shown that the solution is governed by the Wiener-Hopf equation [3]

$$\int_{-\infty}^{\infty} h(\tau - u)R_{y+n}(u)du = R_{y+n,y}(\alpha + \tau), \tau \geq 0 \quad (11)$$

Note that the presence of the constraint $\tau \geq 0$ prevents us from taking the two-sided Laplace transform. The solution to the Wiener-Hopf equation gives instead the linear filter transfer $H(s)$ function as [2], [3], [4]

$$H(s) = \frac{1}{S_{y+n}^+} \left\{ \frac{S_{y+n,y} e^{\alpha s}}{S_{y+n}^-} \right\}^+ \quad (12)$$

where the superscript “+” denotes the causal or positive time part of the signal, “-” denotes the noncausal or negative time part of the signal, and S denotes the spectral density function. The spectral density function is the two-sided Laplace transform of the autocorrelation. The expression requires two-sided Laplace transformation, clipping to remove negative time portions, then one-sided Laplace transformation. This is a series of steps that most students find quite tedious. We now propose an alternative approach.

For a filter of the form (4), the Wiener-Hopf equation becomes

$$\sum_{i=1}^n z_i e^{\lambda_i \tau} \int_{-\infty}^{\infty} e^{-\lambda_i u} R_{y+n}(u) l(\tau - u) du = R_{y+n,y}(\alpha + \tau), \tau \geq 0 \quad (13)$$

Using the Laplace transform version of (2), we rewrite (13) as

$$\sum_{i=1}^n z_i e^{\lambda_i \tau} \mathcal{L} \left[R_{y+n}(u) l(\tau - u) \right]_{s=\lambda_i} = R_{y+n,y}(\alpha + \tau), \tau \geq 0 \quad (14)$$

The solution of the above equation yields the unknown parameters $z_i, \lambda_i, i=1, 2, \dots, n$.

Example 2 [3]

Consider the filtering problem ($\alpha = 0$) with correlation functions

$$R_{y+n}(u) = \delta(u) + e^{-|u|}$$

$$R_{y+n,y}(u) = e^{-|u|}, \alpha = 0$$

The delta function is considered separately and its integration yields

$$\int_{-\infty}^{\infty} h(\tau - u)\delta(u)du = h(\tau)$$

The exponential term requires evaluating the transform of the function shown in Figure 1.

The step function defines the duration of the relevant functions. The transform is

$$\begin{aligned} \mathcal{L} \left[R_{y+n}(u) l(\tau - u) \right] &= \mathcal{L} \left[R_{y+n}(u) - R_{y+n}(u) l(u - \tau) \right] \\ &= \frac{2}{-s^2 + 1} - \frac{e^{-\tau} e^{-s\tau}}{s + 1} \end{aligned}$$

The Wiener-Hopf equation gives

$$e^{-\tau} = \sum_{i=1}^n z_i e^{\lambda_i \tau} \left[1 + \frac{2}{-\lambda_i^2 + 1} - \frac{e^{-\tau} e^{-\lambda_i \tau}}{\lambda_i + 1} \right], \quad \tau \geq 0$$

Equating coefficients of the linearly independent exponential terms, and using the fact that the filter is causal, we note that only one exponential term is needed. We now have

$$1 + \frac{2}{-\lambda_i^2 + 1} = \frac{-\lambda_i^2 + 3}{-\lambda_i^2 + 1} = 0 \Rightarrow \lambda = -\sqrt{3}$$

$$1 = -\frac{z}{\lambda + 1} \Rightarrow z = \sqrt{3} - 1$$

Hence the filter is given by the function

$$h(\tau) = \begin{cases} (\sqrt{3} - 1)e^{-\sqrt{3}\tau}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

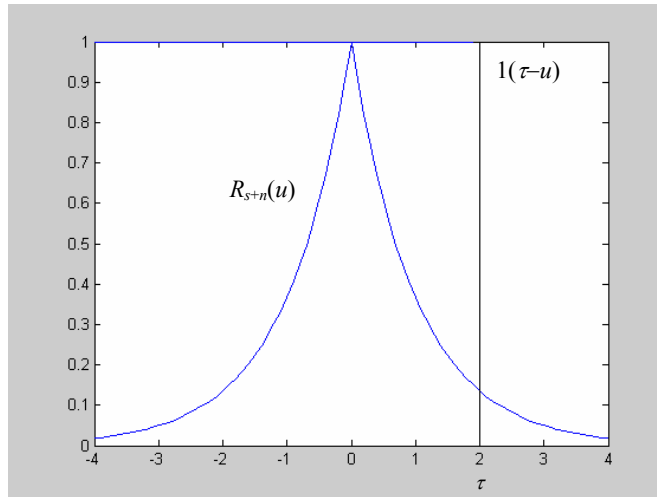


Fig. 1- Plot of the autocorrelation of the filter input and the step function for $\tau = 2$.

VI. Causal Wiener Filter: The Nonstationary Case

In the nonstationary case, the causal Wiener filter is governed by the integral equation [1]

$$\int_0^t h(u, t) R_{y+n}(u - \tau) du = R_{y+n, y}(\alpha + \tau), \quad 0 \leq \tau \leq t \quad (15)$$

A typical solution of the above equation requires differentiation, analytical solution of the resulting differential equation, then substitution into the original integral equation, and equating coefficients. This is already a formidable task. However, in the not uncommon case of a measurement spectral density function having a higher order numerator than its denominator, we also need to add impulses and possibly derivatives of impulses to the

solution. Needless to say, instructors may be reluctant to present this topic to their students because of the long process involved in solving (15). In fact, few texts [3], [8], provide an example involving a nonstationary causal Wiener filter. Our approach makes the solution accessible to the advanced undergraduate or beginning graduate student.

Figure 2 shows an example of an autocorrelation plot delayed by τ as in (15). The plot shows that the autocorrelation function is even when $\tau = 0$, and shows the effect of the delay and clipping for negative arguments. Because the autocorrelation is an even function, we can separate the integral into two parts with limits discerned from Figure 2

$$\int_0^\tau h(u, t)R_{y+n}(u - \tau)du + \int_\tau^t h(u, t)R_{y+n}(u - \tau)du = R_{y+n, y}(\alpha + \tau), \quad 0 \leq \tau \leq t \quad (16)$$

It is well known that a weighting function is realizable if and only if it can be factored as a product of two terms [9] as follows

$$h(t, \tau) = H_1(t)H_2(\tau), \forall t, \tau \quad (17)$$

where H_i , $i = 1, 2$, are matrices of appropriate dimensions. Typically, the weighting function for a causal filter is written in the form

$$h(t, \tau) = \begin{bmatrix} z_1(t) & z_2(t) & \cdots & z_n(t) \end{bmatrix} \begin{bmatrix} e^{\lambda_1 \tau} \\ e^{\lambda_2 \tau} \\ \vdots \\ e^{\lambda_n \tau} \end{bmatrix} \mathbf{1}(\tau) \quad (18)$$

$$= \sum_{i=1}^n z_i(t) e^{\lambda_i \tau} \mathbf{1}(\tau)$$

the equation becomes

$$\sum_{i=1}^n z_i(t) \left\{ \mathcal{L} \left[R_{y+n}(u - \tau) \{ \mathbf{1}(u) - \mathbf{1}(u - \tau) \} \right] + \mathcal{L} \left[R_{y+n}(u - \tau) \{ \mathbf{1}(u - \tau) - \mathbf{1}(u - t) \} \right] \right\}_{s=\lambda_i} \quad (19)$$

$$= R_{y+n, y}(\alpha + \tau), \quad 0 \leq \tau \leq t$$

Example 3 [3]

Consider the nonstationary version of the Wiener filtering problem ($\alpha = 0$) of Example 2. A nonstationary analysis is needed for an optimal Wiener filter over a finite duration t [3] and the coefficients of the impulse response of (4) are now functions of time.

For the processes of Example 2, equation (17) gives

$$\begin{aligned}
R_{y+n,y}(\tau) &= e^{-\tau} \\
&= \sum_{i=1}^n z_i(t) \left\{ \mathcal{L} \left[R_{y+n}(u-\tau) \{1(u) - 1(u-\tau)\} \right] + \mathcal{L} \left[R_{y+n}(u-\tau) \{1(u-\tau) - 1(u-t)\} \right] \right\} \\
&= \sum_{i=1}^n z_i(t) \left[e^{\lambda_i \tau} \left(1 + \frac{2}{-\lambda_i^2 + 1} \right) + \frac{e^{(\lambda_i-1)t} e^{\tau}}{\lambda_i - 1} - \frac{e^{-\tau}}{\lambda_i + 1} \right], \quad 0 \leq \tau \leq t
\end{aligned}$$

Equating coefficients of the linearly independent exponential terms, we now have

$$e^{\lambda_i \tau} : 1 + \frac{2}{-\lambda_i^2 + 1} = \frac{-\lambda_i^2 + 3}{-\lambda_i^2 + 1} = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{3}$$

$$e^{\tau} : \frac{e^{(\lambda_1-1)t}}{\lambda_1 - 1} z_1 + \frac{e^{(\lambda_2-1)t}}{\lambda_2 - 1} z_2 = 0$$

$$e^{-\tau} : -\frac{1}{\lambda_1 + 1} z_1 - \frac{1}{\lambda_2 + 1} z_2 = 1$$

We solve the following equation

$$\begin{bmatrix} \frac{e^{\lambda_1 t}}{\lambda_1 - 1} & \frac{e^{\lambda_2 t}}{\lambda_2 - 1} \\ \frac{1}{\lambda_1 + 1} & \frac{1}{\lambda_2 + 1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence the filter is given by the function

$$h(\tau, t) = \begin{cases} z_1(t) e^{\sqrt{3}\tau} + z_2(t) e^{-\sqrt{3}\tau}, & 0 \leq \tau \leq t \\ 0, & \text{elsewhere} \end{cases}$$

where

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(\sqrt{3}-1)e^{-\sqrt{3}t}}{(\sqrt{3}+2)e^{\sqrt{3}t} - (2-\sqrt{3})e^{-\sqrt{3}t}} \\ \frac{(\sqrt{3}+1)e^{\sqrt{3}t}}{(\sqrt{3}+2)e^{\sqrt{3}t} - (2-\sqrt{3})e^{-\sqrt{3}t}} \end{bmatrix}$$

Note that we obtained the same answer using the method of [3], which is identical to the answer in the original text by Brown [8]. However, our solution using both approaches indicates that one of the coefficients given in [3] has an incorrect numerator. In addition, we can verify the validity of our answer by taking the limit as $t \rightarrow \infty$ to obtain the solution for the stationary causal case. However, the algebra involved in the transform/invert approach is simpler than that of [3],[8].

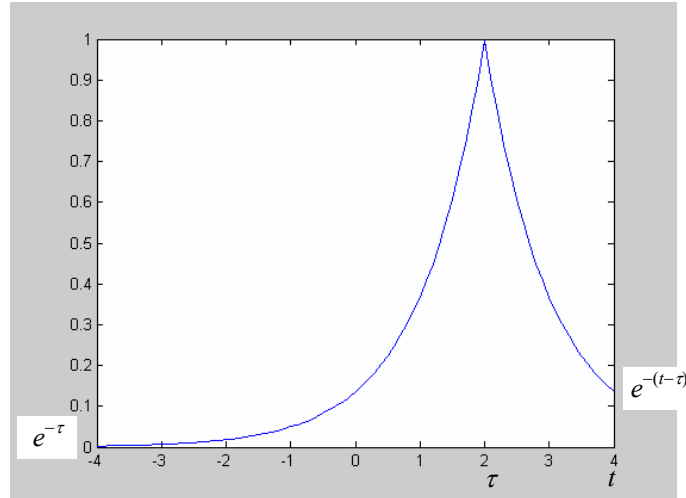


Fig. 2- Plot of the delayed autocorrelation of the filter input for $\tau=2$.

VII. Conclusion

Traditionally, in estimation and statistical signal processing courses, teachers introduced first the Wiener filter and then the Kalman filter, in the chronological order they were derived. One difficulty is the complicated the Wiener-Hopf equation, a Fredholm integral equation, which can be intimidating to engineering students. Alternatively, the Wiener filter is introduced only after the steady state Kalman filter has been understood. The teacher can then explain the Wiener filter as a frequency domain version of the steady-state Kalman filter. This is a simpler approach, but the student does not see the historical development that makes filtering a fascinating field [2], Ch.2], [4]. This paper suggests an approach that makes the Wiener filter less intimidating.

We have tried this approach in an introductory graduate level course on random signals and estimation. Student reaction was quite favorable with all students agreeing that the transform/invert approach was much simpler than the approach of [3].

VIII. References

- [1] N. Wiener, *The Extrapolation, Interpolation and Smoothing of Stationary Time Series*, New York, Wiley, 1949.
- [2] R. S. Mangoubi, *Robust Estimation and Failure Detection*, Springer Verlag, London, 1998.
- [3] R. G. Brown and P. Y. C. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, J. Wiley, NY, 1997.
- [4] T. Kailath, A. H. Sayed, and B. Hassibi, *Linear Estimation*, Prentice Hall, Upper Saddle River, NJ, 2000.

- [5] J. A. Kleppe, “An interesting application of transform theory to the solution of linear systems and certain linear expressions”, *Annual Houston Conference on Circuits, Systems, and Computers*, Houston, TX, 1970.
- [6] A. V. Oppenheim, A. S. Willsky and S. H. Nawab, *Signals and Systems*, Prentice Hall, Upper Saddle River, NJ, 1997.
- [7] K. Ogata. *Modern Control Engineering*, Prentice Hall, Upper Saddle River, NJ, 2002.
- [8] R. G. Brown, *Introduction to Random Signal Analysis and Kalman Filtering*, J. Wiley, NY, 1983.
- [9] W. J. Rugh, *Linear System Theory*, Prentice Hall, Upper Saddle River, NJ, 1996.

Acknowledgement

The authors would like to thank our students, Hyun Cho, Marius Nel and Assem Sonbol, for providing feedback regarding the proposed presentation of the Wiener filter. The authors also acknowledge the reviewers whose comments improved the presentation of this paper.