

Stability Testing for Systems with Polynomial Uncertainty

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Abstract

We develop a new stability tests for systems with one uncertain parameter with polynomial uncertainty structure. The test is derived using the resultant determinant for the real and imaginary parts of the polynomial evaluated on the imaginary axis. The resultant determinant is a function of the uncertain parameter as well as frequency. We evaluate the determinant using a known algorithm then test is for roots using Sturm's theorem. We apply Sturm's test twice: over the allowable range of the uncertain parameter, and for positive angular frequencies. The procedure yields a necessary and sufficient stability condition with polynomial uncertainty structure and one uncertain parameter. We demonstrate the new test using a numerical example.

I. Introduction

The robust stability of uncertain polynomials has been studied and numerous papers and texts. Two excellent texts on the subject discuss the cases of interval polynomials, affine polynomials, and multiaffine polynomials [1], [2]. The main result for the multiaffine case is the mapping theorem of Zadeh and Desoer [3] which characterizes the convex hull of the value set. This provides a sufficient stability condition using the zero exclusion principle [1], [2]. An extension of this theorem due to Barmish and Tempo allows the generation of the value set for a larger class of uncertainty structures. In [4], the authors provide conditions under which the value set of a multiaffine polynomial is equal to its convex hull.

While the exact nature of the value set can be obtained using gridding for polynomials with complex uncertainty structure, this is often impracticable due to its heavy computational burden. This is a significant obstacle to developing necessary and sufficient stability conditions for such uncertainty structures. Zettler and Garloff [5] provide two algorithms using Bernstein expansion for checking the stability of a polynomial in the general case of l uncertain parameters and polynomial uncertainty. While their result is extremely powerful, it is rather complex and a simpler test is needed for the case of a single parameter. A simpler result is available for two or three parameters using symbolic manipulation but it is not suitable for higher order polynomials in the uncertain parameter [6]. To the author's knowledge, there is currently no simple stability test for the polynomial uncertainty structure, other than using gridding. The main aim of this paper is to provide a necessary and sufficient stability test for the polynomial uncertainty case.

To develop our test, we consider the literature for linear dependence on a single parameter [7], [8], [9]. The best known test is that due to Bialas [7] and its proof using the resultant determinant suggests part of the proof of our result. We also use Sturm's theorem to test the existence of polynomial roots in a given interval as in [8]. The resultant determinant for polynomial uncertainty structure is a two-variable polynomial in the uncertain parameter as well as the angular frequency. We use the algorithm of [10] to obtain the determinant then use Sturm's theorem twice to test for the existence of a root in the feasible ranges of frequency and parameter values.

To emphasize the importance of the results of this paper, we refer the reader to the work of Farouki et al. [11]. Their work shows characterize the product of sets in the complex plane, including straight lines. In the multiaffine case with two uncertain parameters, it is well known that the boundary of the value set can include parabolic segments [1], p. 164]. A parametric representation of this curve is quadratic in a single parameter. Multiaffine uncertainty with more uncertain parameters yields higher order curves. Thus there is a need for test for robust stability in the case of polynomial uncertainty.

The remainder of the paper is organized as follows. In Section II, we review the main results needed to develop the new stability test and state the main result of the paper. In Section III, we give examples that demonstrate the use of the new result. Section IV concludes the paper and discusses ideas for future work.

II. Methodology

Consider the uncertain polynomial

$$p(s, q) = \sum_{i=0}^l p_i(s) q^i \quad (1)$$

with known polynomials

$$p_i(s) = \sum_{j=0}^{n_i} a_{ij} s^j, i = 0, 1, \dots, l \quad (2)$$

where a_{ij} are constant coefficients and $n_i \leq n$.

Without loss of generality, we assume that the uncertain parameter is bounded such that

$$q \in [0, 1] \quad (3)$$

We also assume $n_0 = n$ and a_{0n} so that degree dropping is excluded.

At any given angular frequency, the known polynomials of (2) become

$$\begin{aligned} p_i(j\mathbf{w}) &= g_i(-\mathbf{w}^2) + j\mathbf{w}h_i(-\mathbf{w}^2) \\ &= \sum_{j=0}^{\lfloor \frac{n_i-1}{2} \rfloor} a_{i,2j} (-\mathbf{w}^2)^j + j\mathbf{w} \sum_{j=0}^{\lfloor \frac{n_i-1}{2} \rfloor} a_{i,2j+1} (-\mathbf{w}^2)^j, i = 0, 1, \dots, l \end{aligned} \quad (4)$$

where $\lfloor \bullet \rfloor$ denotes the floor and $\lceil \bullet \rceil$ denotes the ceiling.

Substituting in (1) gives the expression

$$\begin{aligned}
p(j\mathbf{w}, q) &= \sum_{i=0}^l [g_i(-\mathbf{w}^2) + j\mathbf{w}h_i(-\mathbf{w}^2)]q^i \\
&= \sum_{i=0}^l \left[\sum_{j=0}^{\lfloor \frac{n_i-1}{2} \rfloor} a_{i,2j}(-\mathbf{w}^2)^j + j\mathbf{w} \sum_{j=0}^{\lfloor \frac{n_i-1}{2} \rfloor} a_{i,2j+1}(-\mathbf{w}^2)^j \right] q^i
\end{aligned} \tag{5}$$

Excluding the zero angular frequency, the polynomial is zero if its real and imaginary parts are both zero. It is convenient to use the variable $x := -\mathbf{w}^2$. We therefore seek common zeros of the two polynomials

$$\begin{aligned}
g(x, q) &= \sum_{i=0}^l g_i(x)q^i \\
&= \sum_{i=0}^l \left[\sum_{j=0}^{\lfloor \frac{n_i-1}{2} \rfloor} a_{i,2j}x^j \right] q^i = \sum_{j=0}^{\lfloor \frac{n_l-1}{2} \rfloor} x^j \sum_{i=0}^l a_{i,2j}q^i = 0
\end{aligned} \tag{6}$$

$$\begin{aligned}
h(x, q) &= \sum_{i=0}^l h_i(x)q^i \\
&= \sum_{i=0}^l \left[\sum_{j=0}^{\lfloor \frac{n_i-1}{2} \rfloor} a_{i,2j+1}x^j \right] q^i = \sum_{j=0}^{\lfloor \frac{n_l-1}{2} \rfloor} x^j \sum_{i=0}^l a_{i,2j+1}q^i = 0
\end{aligned} \tag{7}$$

In particular, we seek common zeros of the polynomial in the region $\{\mathbf{w} > 0, q \in [0, 1]\}$.

Two polynomials have a common zero if their resultant determinant is zero as discussed in Appendix B of [1]. For two polynomials

$$\begin{aligned}
a(x) &= a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\
b(x) &= b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0
\end{aligned} \tag{8}$$

the resultant matrix is

$$R(a, b) = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_n & & a_1 & a_0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & & \vdots \\ 0 & 0 & \cdots & a_n & a_{n-1} & \cdots & a_0 & \\ b_m & b_{m-1} & & b_1 & b_0 & 0 & \cdots & 0 \\ & b_m & \cdots & & b_1 & b_0 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & b_m & b_{m-1} & \cdots & b_0 & \end{bmatrix} \quad (9)$$

The matrix is square of order $n+m$ with the first m rows including coefficients of $a(x)$ and the next n rows including coefficients of $b(x)$. Theorem 1 provides a necessary and sufficient condition for two polynomials $a(x)$ and $b(x)$ to have a common zero.

Theorem 1 [1]

Two polynomials $a(x)$ and $b(x)$ have a common zero if and only if

$$\det[R(a, b)] = 0 \quad (10)$$

Considering the polynomials as function of x with coefficients depending on q , the resultant matrix corresponding to (6) and (7) can be written in the form

$$R_q(g, h) = \sum_{i=0}^l R_i q^i \quad (11)$$

The determinant of the resultant can be obtained by the algorithm of [10] for the inversion of polynomial matrices. If the determinant is nonzero for all q values in the feasible range of (3) then we conclude that the polynomial is stable. If the determinant is zero for a feasible q value, we need to consider the two polynomials as functions of q with coefficients depending on x .

$$R_x(g, h) = \sum_{j=0}^{\left\lceil \frac{n+1}{2} \right\rceil} R_j x^j \quad (12)$$

The main result of this paper follows from the above discussion.

Theorem 2

A polynomial $p(s, q)$ with the uncertainty structure of (1), and with stable component $p(s, 0) = p_0(s)$, is stable if and only if

$$\text{roots}[R_q(g, h)] \notin [0, 1] \quad (13)$$

$$\text{roots}[R_x(g, h)] \notin \mathbb{R}^- \quad (14)$$

Proof

The proof of the theorem follows from the discussion and with the use of the zero exclusion principle and the boundary crossing theorem [1].

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To test the existence of roots of a polynomial in a given range, we use Sturm's theorem.

Theorem 3

A polynomial $f(x)$ has a number of roots in a real interval $[low, high]$ equal to the difference between the sign changes S at *high* and the sign changes S at *low*, where S is the number of sign changes over the set of polynomials $f_i, i=1,2,3,4$, with

$$f_1 = f, f_2 = \frac{df}{dx}, f_i = \frac{Q_i}{f_{i+1}} + f_{i+2}, i = 1, 2, \dots, n-2 \quad (15)$$

Based on the main result of Theorem 2, we have the following procedure for robust stability testing. We exclude the zero frequency from our search since its test is trivial.

Procedure

1. Separate each polynomial $p_i(j\boldsymbol{w}), i=0,1,\dots,l$, into its real and imaginary parts.
2. Obtain the resultant determinant as a polynomial in q and test for roots in $[0,1]$ using Sturm's theorem. Absence of roots indicates a robustly stable polynomial.
3. If a real root is found in the interval $[0,1]$, obtain the resultant determinant as a polynomial in x and test for roots in the interval $[-\infty,0)$ using Sturm's theorem. If no roots exist, the polynomial is robustly stable. If roots are found, the polynomial is not robustly stable.

We now demonstrate the application of the new theorem through an example.

III. Example

We consider a polynomial of the form (1) with $l=2$ and with

$$p_0(s) = s^5 + 2s^4 + 3s^3 + 4s^2 + 0.001s + 4$$

$$p_1(s) = 4s^5 - 7s^4 + s^3 + 5s^2 + s - 1$$

$$p_2(s) = s^5 + 7s^4 + 7s^3 + 10s^2 + 5s + 2$$

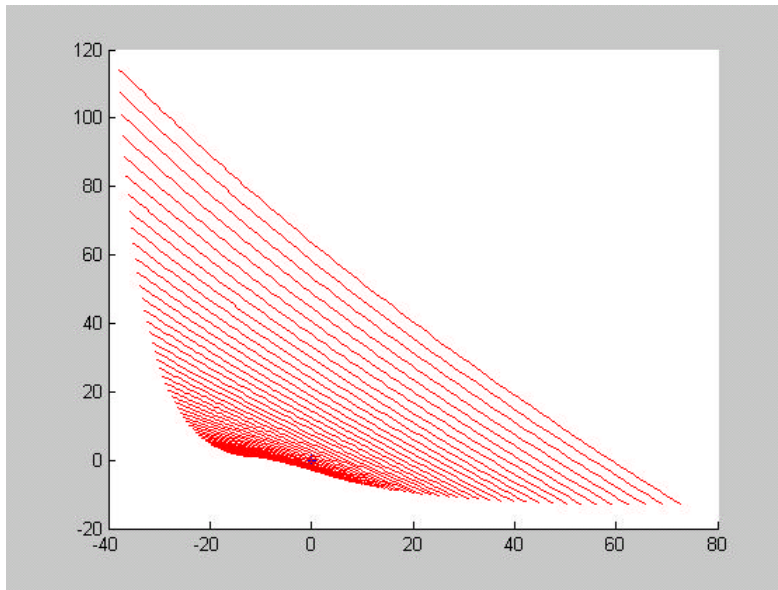


Figure 1. Value sets for the polynomial for a frequency range [2,3] rad/s.

The stability test detects a feasible root for both $q \in [0,1]$ and $x \in [-10^{36},0]$ or equivalently $\mathbf{w} \in [0, 10^{18}]$. We conclude that the system is unstable. The result is verified using gridding to compute the value set. Furthermore, on computing the roots of the resultants we detect boundary crossing at $q = 0.03148$ and $\mathbf{w} = 1.4587$ rad/s. Identical results are obtained using gridding as shown in Figure 1.

References

- [1] J. Ackermann, A. Bartlett, D. Kaesbauer, W. Sienel, and R. Steinhauser. *Robust Control: Systems with Uncertain Physical Parameters*. London: Springer-Verlag, 1993.
- [2] B. R. Barmish. *New Tools for Robustness of Linear Systems*. New York: Macmillan Publishing, 1994.
- [3] L. A. Zadeh and C. A. Desoer, *Linear System Theory*. New York: McGraw Hill, 1963.
- [4] B. D. O. Anderson, F. Kraus, M. Mansour, and S. Dasgupta. "Easily testable sufficient conditions for the robust stability of systems with multilinear parameter dependence", *Automatica*, vol. 31, No. 1, 25-40, 1995.
- [5] M. Zettler and J. Garloff, "Robustness analysis of polynomials with polynomial parameter dependency using Bernstein expansion", *IEEE Trans. Automat. Contr.*, vol. 43, pp. 425-431, 1998.
- [6] D. Kaesbauer, "On robust stability of polynomials with polynomial parameter dependency: two/three parameter cases", *Automatica*, vol. 29, pp. 215-217, 1993.
- [7] S. Bialias, "A Necessary and Sufficient Condition for the Stability of Convex Combinations of Stable Polynomials or Matrices", *Bulletin of the Polish Academy of Sciences*, vol. 33, pp. 473-480, 1985.
- [8] H. Bougerra, B. C. Chang, H. H. Yeh and S. S. Banda, "Fast Stability Checking for the Convex Combination of Stable Polynomials", *IEEE Trans. Automat. Contr.*, vol. 35, pp. 586-588, 1990.

- [9] V. Blondel, "A Note on Convex Combinations of Polynomials", *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1690-1691, 1996.
- [10] M. Buslowicz, "Inversion of polynomial matrices", *Int. J. Control*, vol. 33, No. 5, pp. 977-984, 1980.
- [11] R. T. Farouki, H. P. Moon, and B. Ravani, "Minkowski geometric algebra of complex sets", *Geometriae Dedicata*. To appear. Online at <http://mae.ucdavis.edu/~farouki/>.