

# **Timely Robust Fault Detection for Multirate Linear Systems**

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## **Abstract**

This paper presents a fault detection and isolation scheme for multirate systems with a fast input sampling rate and slower output sampling rates. We design a separate observer for each set of simultaneous measurements with the observer operating at their sampling rate. We use an unknown input observer design to allow state estimation in the presence of disturbances and modelling errors. The observer allows us to estimate the system state and obtain a residual vector to be used in fault detection. Furthermore, we are able to use single-rate methodologies for fault diagnosis. We provide necessary and sufficient conditions for the existence of the observer and the detection of the fault vector. An example is given to illustrate the new fault detection approach and another to demonstrate fault isolation.

## **1. Introduction**

Multirate sampling is required in many industrial applications and their modelling and control have received considerable attention in the literature [5]. In the early literature,  $z$ -transform methods were adopted for modelling multirate sampled systems [13]. An important contribution is the development of a time-invariant model or lift representation of multirate sampled systems [14]. More recent literature has used state-space models for the modelling analysis and control systems design of multirate systems [1, 4, 5, 10, 16].

An important problem that has received little attention in the literature on multirate control is the problem of fault detection [3, 9, 11, 15]. As applied to multirate systems, the problem has only been considered in two papers [6], [7]. Both papers use a lift reformulation to show that single rate techniques for fault detection can provide robust fault detection for multirate systems. The first paper [6] uses the parity approach of Chow and Willsky [3], and the second [7] uses an observer-based approach. The two papers were exploratory in nature and served to suggest fault

detection for multirate systems as an interesting research topic. However, the majority of applications involving multirate systems require more sophisticated schemes specifically designed to suit their structure. The main drawback of [6] and [7] is that they can only detect a fault at the end of a repetition period for the multirate system. In many cases this is an excessive delay relative to the fast control loops of the system. A more general multirate fault detection scheme must utilize measurements as they become available rather than wait for the end of the repetition period.

This paper presents a new fault detection scheme specifically designed for multirate systems. We restrict our study to the case of fast input sampling and slower output sampling with period equal to a given multiple of the input sampling period. We assume that we can summarize all uncertainties of a system as disturbances with known input matrix. We use the state equation for the unique input-sampling rate together with each set of synchronous measurement to obtain a time-invariant system representation. This representation is then used to design an observer utilizing the measurements as they become available. The observer is used to estimate the observable fault-coupled partial state of the system using the available measurements. We repeat the process for other synchronous measurement sets to obtain a bank of observers running at different rates and capable of detecting both actuator and sensor faults. For each observer, we use the unknown input observer scheme of Hou and Muller [11] to eliminate the effect of disturbances, and then retain the fault-coupled observable subsystem for use in residual generator design. A preliminary short version of the paper was published in conference proceedings [8].

Note that our observer design is similar to that of Colaneri [17] for periodic systems where a standard Luenberger observer is used in place of a Hou-Muller observer. In addition, [17]

completes the observer design in one step whereas we separate the designs for each subsystem. The separation allows us to only estimate the partial state of interest in fault detection. It also allows fault isolation and diagnosis using single-rate techniques. Other observer designs for multirate systems (see for example [18] and references therein) significantly complicate fault diagnosis and/or disturbance rejection.

To illustrate our scheme, we consider the case of a system with input sampling period  $T_0$  and output sampling periods  $T_i, i = 1, 2$ , as shown in Figure 1. For this example, two observers are needed: one with sampling period  $T_1$  and the other with sampling period  $T_2$ . The observers are used to provide estimates of the outputs  $\hat{y}_i$ , which are then compared to the output measurement  $y_i$ , to obtain two residual vectors  $r_i, i=1, 2$ . The residual vectors are used for fault detection and isolation as in the single rate case.

The paper is organized as follows. In Section 2, we review the model of the class of multirate systems addressed in this paper. In Section 3, we describe the new unknown input observer and fault detection scheme. In Section 4, we provide necessary and sufficient conditions for fault coupling and fault detection. In Section 5, we apply the new scheme for fault detection to an illustrative example. Conclusions are given in Section 6.

Throughout the paper, we use the following notation:

$x$	state vector
$y$	output vector
$d$	disturbance vector.
$f_a (f_s)$	actuator (sensor) fault vector.
$A$ or $F$	state matrix.
$B$ or $G$	input matrix.
$C$	output matrix.
$\mathbb{C}$	set of complex numbers
$E$	direct transmission matrix for a sensor fault vector.

superscript $u, d, f$	refers to the control, disturbance, and fault, respectively.
matrix subscript $c$	component matrix of the output matrix $C$
subscript $d$	refers to a discrete-time system.
matrix subscript $g$	component matrix of the input matrix $G$
subscript $h$	refers to the system after a disturbance eliminating transformation.
subscript $i$	refers to the $i$ th subsystem.
superscript $*$	denotes the conjugate transpose.
superscript $\hat{\phantom{x}}$	denotes an estimate

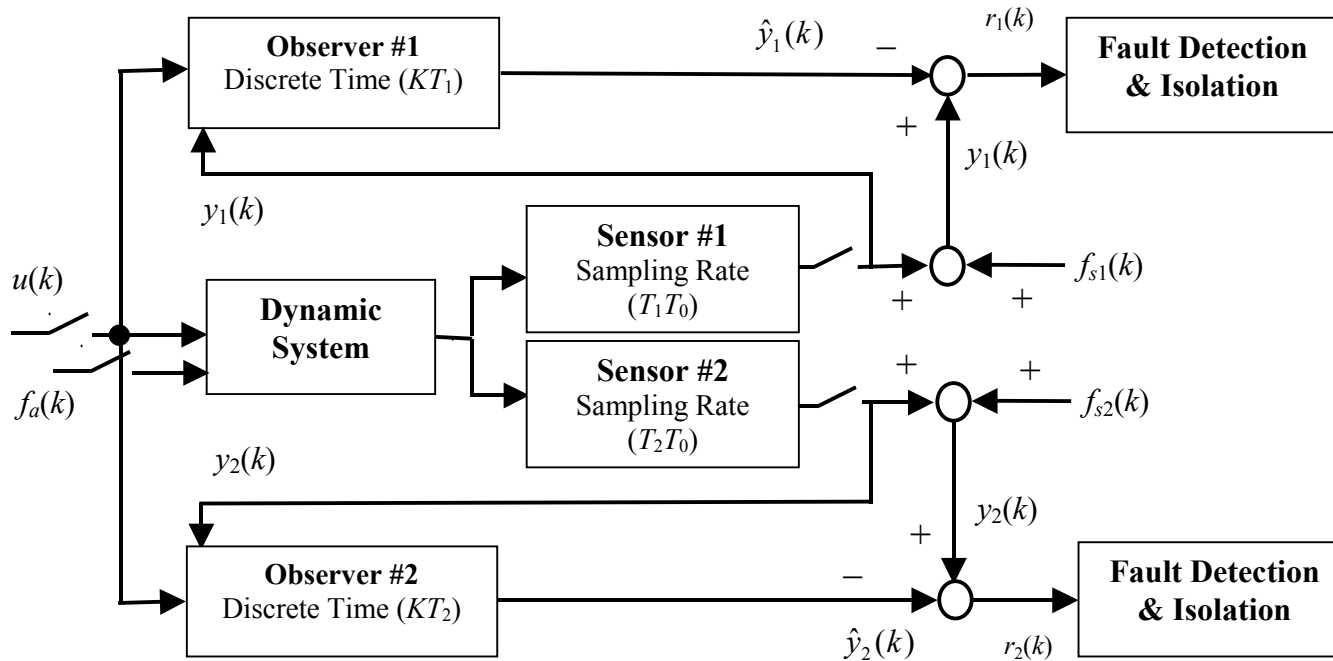


Figure 1. Illustration of multirate fault detection and isolation for a two-rate system.

## 2. Multirate Linear System

Consider a linear time invariant system in the continuous form

$$\dot{x}(t) = Ax(t) + B^u u(t) + B^d d(t) + B^f f_a(t) \quad (1a)$$

$$y(t) = Cx(t) + Ef_s(t) \quad (1b)$$

where  $A \in R^{n,n}$ ,  $B^u \in R^{n,p}$ ,  $B^d \in R^{n,q}$ ,  $B^f \in R^{n,r}$ ,  $C \in R^{m,n}$ , and  $E \in R^{m,r_s}$ ,  $x(t)$  is an  $n$  by 1 state vector,  $u(t)$  is a  $p$  by 1 vector of known inputs,  $d(t)$  is a  $q$  by 1 unknown disturbance vector,  $f_a(t)$  is a  $r$  by 1 actuator fault vector, and  $f_s(t)$  is a  $r_s$  by 1 sensor fault vector.

We assume that the disturbance and fault vectors are approximately constant over the short input sampling period  $T_0$ . If the output sampling period is equal to the unique input sampling period  $T_0$ , we have the discrete-time linear time-invariant model

$$x(k+1) = A_d x(k) + B_d^u u(k) + B_d^d d(k) + B_d^f f_a(k) \quad (2a)$$

$$y(k) = Cx(k) + Ef_s(k) \quad (2b)$$

where  $x(k), u(k), d(k), f_a(k), f_s(k)$  and the corresponding matrices have the same significance and order as their continuous counterparts.

Assume that the output vector includes subsets  $y_i(k), i = 1, \dots, n_m$ , and that the  $i$ th output vector is updated every  $T_i T_0$  s where  $T_i, i = 1, \dots, n_m$ , are positive integers. The  $i$ th output vector has  $m_i$  components such that  $m = \sum_{i=1}^{n_m} m_i$ . We require  $m_i > 1, i = 1, 2, \dots, n_m$  for fault detection and disturbance rejection.

Consider the solution of the discrete-time state equation (2a). For any positive integer  $l$ , the solution is

$$x(kT_i + l) = A_d^l x(kT_i) + \sum_{j=kT_i}^{kT_i+l-1} A_d^{kT_i+l-1-j} B_d^u u(j) + \sum_{j=kT_i}^{kT_i+l-1} A_d^{kT_i+l-1-j} B_d^d d(j) + \sum_{j=kT_i}^{kT_i+l-1} A_d^{kT_i+l-1-j} B_d^f f_a(j) \quad (3)$$

We first rewrite the solution for  $l = T_i$  in the form

$$\begin{aligned} x(kT_i + T_i) &= A_d^{T_i} x(kT_i) + \begin{bmatrix} A_d^{T_i-1} B_d^u & \cdots & A_d B_d^u & B_d^u \end{bmatrix} \begin{bmatrix} u(kT_i) \\ \vdots \\ u(kT_i + T_i - 2) \\ u(kT_i + T_i - 1) \end{bmatrix} \\ &+ \begin{bmatrix} A_d^{T_i-1} B_d^d & \cdots & A_d B_d^d & B_d^d \end{bmatrix} \begin{bmatrix} d(kT_i) \\ \vdots \\ d(kT_i + T_i - 2) \\ d(kT_i + T_i - 1) \end{bmatrix} \\ &+ \begin{bmatrix} A_d^{T_i-1} B_d^f & \cdots & A_d B_d^f & B_d^f \end{bmatrix} \begin{bmatrix} f_a(kT_i) \\ \vdots \\ f_a(kT_i + T_i - 2) \\ f_a(kT_i + T_i - 1) \end{bmatrix} \end{aligned} \quad (4)$$

Let  $k$  be a new discrete-time variable and define

$$\begin{aligned} \bar{x}_i(k) &:= x(kT_i) \\ \bar{u}_i(k) &:= \text{col}\{u(kT_i), u(kT_i + 1), \dots, u(kT_i + T_i - 1)\} \\ \bar{f}_{ai}(k) &:= \text{col}\{f_a(kT_i), f_a(kT_i + 1), \dots, f_a(kT_i + T_i - 1)\} \\ \bar{d}_i(k) &:= \text{col}\{d(kT_i), d(kT_i + 1), \dots, d(kT_i + T_i - 1)\} \\ \bar{f}_{si}(k) &:= \text{col}\{f_s(kT_i), f_s(kT_i + 1), \dots, f_s(kT_i + T_i - 1)\} \end{aligned} \quad (5)$$

The  $i$ th system is defined by

$$\bar{x}_i(k+1) = F_i \bar{x}_i(k) + G_i^u \bar{u}_i(k) + G_i^d \bar{d}_i(k) + G_i^f \bar{f}_{ai}(k) \quad (6a)$$

$$y_i(k) = C_i \bar{x}_i(k) + E_i \bar{f}_{si}(k) \quad (6b)$$

where

$$F_i = A_d^{T_i} \quad (7c)$$

$$G_i^l = [G_{i,0}^l \quad G_{i,1}^l \quad \cdots \quad G_{i,T_i-1}^l] \quad (7b)$$

$$G_{i,j}^l = A_d^{T_i-j-1} B_d^l, j = 0, \dots, T_i - 1, l \in \{u, d, f\}$$

### 3. State Observer and Fault Detection

We construct a state observer for the observable subspace of the time invariant  $i$ th system of equations (6-7). We first define the disturbance free-subsystem as follows.

#### Definition 1: Disturbance-Free Subsystem

The disturbance-free subsystem of (6) is the subsystem whose state equations do not include the disturbance input explicitly.

The disturbance-free subsystem is obtained via the approach of Hou and Muller [11]. Because our goal is fault detection and not state estimation per se, we need not obtain an estimate of the partial state affected by the disturbance. Note that fault information contained in the disturbance-coupled partial state is retained in the dynamics of the disturbance-free partial state.

To obtain the Hou and Muller transformation we use the singular value decomposition of the disturbance input matrix

$$G_i^d = U_{gi} \begin{bmatrix} \Sigma_{gi} & 0_{g_i, n-g_i} \\ 0_{n-g_i, g_i} & 0_{n-g_i, n-g_i} \end{bmatrix} V_{gi}^* \quad (8)$$

where  $g_i$  is the rank of  $G_i^d$ . We define the similarity transformation

$$\bar{x}_i^h(k) = U_{gi}^* \bar{x}_i(k) \Leftrightarrow \bar{x}_i(k) = U_{gi} \bar{x}_i^h(k) \quad (9)$$

where the superscript  $h$  refers to the Hou-Muller decomposition. We transform the system to

$$\begin{bmatrix} \bar{x}_{i1}^h(k+1) \\ \bar{x}_{i2}^h(k+1) \end{bmatrix} = \begin{bmatrix} F_{i1} & F_{i2} \\ F_{i3} & F_{i4} \end{bmatrix} \begin{bmatrix} \bar{x}_{i1}^h(k) \\ \bar{x}_{i2}^h(k) \end{bmatrix} + \begin{bmatrix} G_{i1}^u \\ G_{i2}^u \end{bmatrix} \bar{u}_i(k) + \begin{bmatrix} G_{i1}^d \\ 0 \end{bmatrix} \bar{d}_i(k) + \begin{bmatrix} G_{i1}^f \\ G_{i2}^f \end{bmatrix} \bar{f}_{ai}(k) \quad (10a)$$

$$\begin{aligned} y_i(k) &= C_i U_{gi} \bar{x}_i^h(k) + E_i f_{si}(k) \\ &= \begin{bmatrix} C_{i1} & C_{i2} \end{bmatrix} \begin{bmatrix} \bar{x}_{i1}^h(k) \\ \bar{x}_{i2}^h(k) \end{bmatrix} + E_i f_{si}(k) \end{aligned} \quad (10b)$$

where  $\bar{x}_{i1}^h(k)$  is the disturbance-coupled partial state and  $\bar{x}_{i2}^h(k)$  is the disturbance-free partial state.

We obtain the singular value decomposition

$$C_{i1} = U_{ci} \begin{bmatrix} \Sigma_{ci} \\ \mathbf{0} \end{bmatrix} V_{ci}^* \quad (11)$$

The unitary matrix  $U_{ci}^*$  is partitioned as

$$U_{ci}^* = \begin{bmatrix} U_{ci1}^* \\ U_{ci2}^* \end{bmatrix} \quad (12)$$

We then transform the output using the unitary matrix to

$$\begin{aligned} \begin{bmatrix} y_{i1} \\ y_{i2} \end{bmatrix} &= \begin{bmatrix} U_{ci1}^* y_i \\ U_{ci2}^* y_i \end{bmatrix} \\ &= \begin{bmatrix} \Sigma_{ci} V_{ci}^* & U_{ci1}^* C_{i2} \\ \mathbf{0} & U_{ci2}^* C_{i2} \end{bmatrix} \begin{bmatrix} \bar{x}_{i1}^h(k) \\ \bar{x}_{i2}^h(k) \end{bmatrix} + \begin{bmatrix} U_{ci1}^* E_i f_{si}(k) \\ U_{ci2}^* E_i f_{si}(k) \end{bmatrix} \end{aligned} \quad (13)$$

We solve for the disturbance-coupled partial states

$$\bar{x}_{i1}^h(k) = V_{ci} \Sigma_{ci}^{-1} \left\{ y_{i1} - U_{ci1}^* \left( C_{i2} \bar{x}_{i2}^h(k) + E_i f_{si}(k) \right) \right\} \quad (14)$$

Since we are only interested in estimating the disturbance-free partial state  $\bar{x}_{h2}(k)$ , the system of interest is

$$\bar{x}_{i2}^h(k+1) = F_{ih} \bar{x}_{i2}^h(k) + G_{ih}^u \bar{u}_i(k) + G_{ih}^y y_{i1}(k) + G_{ih}^f \bar{f}_{hi}(k) \quad (15a)$$

$$y_{i2}(k) = C_{ih} \bar{x}_{i2}^h(k) + E_{ih} \bar{f}_{hi}(k) \quad (15b)$$

where

$$\begin{aligned} F_{ih} &= F_{i4} - F_{i3} V_c \Sigma_c^{-1} U_{c1}^* C_{i2} \\ G_{ih}^y &= F_{i3} V_c \Sigma_c^{-1} \\ G_{ih}^f &= \begin{bmatrix} G_{i2}^f & | & -F_{i3} V_c \Sigma_c^{-1} U_{c1}^* E_i \end{bmatrix} \\ \bar{f}_{hi}(k) &= \begin{bmatrix} \bar{f}_{ai}(k) \\ \bar{f}_{si}(k) \end{bmatrix} \end{aligned} \quad (16)$$

$$C_{ih} = U_{ci2}^* C_{i2}$$

$$E_{ih} = U_{ci2}^* E_i$$

The disturbance-free subsystem may not be completely observable. We must therefore transform the system to observability canonical form [2] and separate the observable subsystem to estimate its state. Furthermore, we need only estimate the fault coupled partial state since the remainder of the state vector is of no use in fault detection. In the presence of sensor and actuator faults, we have a partial state coupled to one or to both types of faults. The following definition makes this notion more precise.

**Definition 2. Fault-Coupled System**

A system with state matrix  $F$  and fault influence matrix  $G^f$  is fault-coupled if it is controllable with the fault as input.

**Definition 3. Fault-Decoupled Mode**

An eigenvalue of  $F$  is a fault-decoupled mode if it is uncontrollable with the fault as input.

Transforming the disturbance-decoupled system to observability canonical form yields the decomposition

$$\begin{bmatrix} \bar{x}_{io}(k+1) \\ \bar{x}_{iu}(k+1) \end{bmatrix} = \begin{bmatrix} F_{io} & 0 \\ F_{iuo} & F_{iu} \end{bmatrix} \begin{bmatrix} \bar{x}_{io}(k) \\ \bar{x}_{iu}(k) \end{bmatrix} + \begin{bmatrix} G_{io}^u \\ G_{iu}^u \end{bmatrix} \bar{u}_i(k) + \begin{bmatrix} G_{io}^y \\ G_{iu}^y \end{bmatrix} y_{i1}(k) + \begin{bmatrix} G_{io}^f \\ G_{iu}^f \end{bmatrix} \bar{f}_{hi}(k) \quad (17a)$$

$$y_{i2}(k) = \begin{bmatrix} C_{io1} & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{io}(k) \\ \bar{x}_{iu}(k) \end{bmatrix} + E_{ih} \bar{f}_{hi}(k) \quad (17b)$$

We now separate the observable disturbance-free subsystem

$$\bar{x}_{io}(k+1) = F_{io} \bar{x}_{io}(k) + G_{io}^u \bar{u}_i(k) + G_{io}^y y_{i1}(k) + G_{io}^f \bar{f}_{hi}(k) \quad (18a)$$

$$y_{i2}(k) = C_{io1} \bar{x}_{io}(k) + E_{ih} \bar{f}_{hi}(k) \quad (18b)$$

We next obtain the fault-coupled subsystem via another similarity transformation. The transformation is discussed further in Section 4. The fault-decoupled subsystem is given by

$$z_i(k+1) = A_{zi} z_i(k) + B_i^u \bar{u}_i(k) + B_i^y y_{i1}(k) + B_i^f \bar{f}_{hi}(k) \quad (19a)$$

$$y_{i2}(k) = C_{zi} z_i(k) + E_{ih} \bar{f}_{hi}(k) \quad (19b)$$

The observer dynamics are given by

$$\begin{aligned}\hat{z}_i(k+1) &= A_{z_i}\hat{z}_i(k) + B_i^u\bar{u}(k) + B_i^y y_{i1}(k) + L[y_{i2}(k) - C_{z_i}\hat{z}_i(k)] \\ &= [A_{z_i} - LC_{z_i}]\hat{z}_i(k) + B_i^u\bar{u}(k) + B_i^y y_{i1}(k) + Ly_{i2}(k)\end{aligned}\quad (20a)$$

$$\hat{y}_{i2}(k) = C_{z_i}\hat{z}_i(k) \quad (20b)$$

The state estimation error is

$$e_i(k) = z_i(k) - \hat{z}_i(k) \quad (21)$$

The error is known to converge to zero if the eigenvalues of the matrix

$$A_{oi} = A_{z_i} - LC_{z_i} \quad (22)$$

are assigned in appropriate subregions of the unit circle.

A residual vector is generated from the difference between the actual and estimated measurements

$$\varepsilon(k) = W(y_i(k) - \hat{y}_i(k)) \quad (23)$$

$$\varepsilon_i(k) = WCe_i(k) + WB_i^f f_{hi}(k) \quad (24)$$

where  $W$  is a weighting matrix that can be selected for fault isolation.

#### 4. Theoretical Results

In this section, we provide some conditions for fault detection using the observer of Section 3. To simplify the notation, we drop the subscript  $i$  even though the results are understood to apply to the  $i$ th subsystem,  $i = 1, 2, \dots, n_i$ . We begin by stating conditions that apply in the case of a negligible disturbance vector.

##### **Theorem 1. Fault-Coupled System**

A system with state matrix  $F$  and fault influence matrix  $G^f$  is fault coupled if and only if one of the following two conditions holds

(i) The fault controllability matrix

$$\mathcal{E}_f = [G^f \mid FG^f \mid \cdots \mid F^{n-1}G^f] \quad (25)$$

is full rank.

(ii) Each left eigenvectors  $w'$  of  $F$  satisfies the condition

$$w'G^f \neq 0$$

**Proof**

The result follows from standard controllability results and Definitions 2 and 3. ♦

For a multirate system, the pair  $(F, G^f)$  corresponding to the  $i$ th system is given by

$$(F, G^f) = (A_d^{T_i-1}, [A_d^{T_i-1}B_d^f, \cdots, A_d B_d^f, B_d^f]) \quad (26)$$

For this pair, we have the following theorem.

**Theorem 2**

The pair  $(F, G^f)$  is fault-coupled if the pair  $(A_d, B_d^f)$  is fault-coupled. Furthermore, if  $\lambda_j$  is a fault-decoupled mode of  $(A_d, B_d^f)$ , then  $\lambda_j^{T_i}$  is a fault-decoupled mode of  $(F, G^f)$ .

**Proof**

We prove the result by direct application of the eigenvector test using the fact that  $(\lambda_j^{T_i}, w'_j)$  is a left eigenpair of  $F$  if  $(\lambda_j, w'_j)$  is a left eigenpair of  $A_d$ . ♦

The multirate fault detection scheme of Section 3 uses the observable subsystem. The following observability result is useful.

**Theorem 3**

The pair  $(F, C)$  is observable if the pair  $(A_d, C)$  is observable. Furthermore,  $\lambda_j^{T_i}$  is an unobservable mode  $(F, G^f)$  if  $\lambda_j$  is an unobservable mode of  $(A_d, C)$ .

**Proof**

The proof is by direct application of the eigenvector test using the fact that  $(\lambda_j^{T_i}, v)$  is a right eigenpair of  $F$  if  $(\lambda_j, v)$  is a right eigenpair of  $A$ . From the definition of an eigenvector

$$A_d v_j = \lambda_j v_j \Rightarrow A_d^{T_i} v_j = \lambda_j^{T_i} v_j$$

The  $j$ th mode is unobservable if and only if

$$C v_j = 0$$

◆

Next, we state a well known condition for the existence of the Hou-Muller observer [11]. The result applies to the quadruple  $(F, G^d, C, E)$  with  $F \in R^{n,n}, G^d \in R^{n,q}, C \in R^{m,n}, E \in R^{m,r_s}$ .

**Theorem 4**

If  $\text{rank}(CG^d) = \text{rank}(G^d) = q$ , then the following statements are equivalent

a) The pair  $(F_h, C_h)$  is detectable.

$$\text{b) } \text{rank} \begin{bmatrix} zI_n - F & G^d \\ C & 0_{m,q} \end{bmatrix} = n + q$$

$$\forall z \in \mathbb{C}, |z| \geq 1$$

**Remark**

The original result due to Hou and Muller [11] is for a continuous time system. However, the proof for the discrete time case is identical with the complex variable  $s$  replaced by the complex

variable  $z$ . In addition, the detectability of the discrete systems requires that the unobservable modes of the pair  $(F, G^d)$  must lie outside the unit circle as stated in condition (b) of Theorem 4.

Although the literature clearly indicates the need for fault coupling in fault detection, no result is available to unequivocally test this condition. Hou and Muller [11] point out that the fault influence matrix in the observable subsystem must be nonzero for successful fault detection. However, they do not discuss the concept of the fault-coupled subsystem and do not provide conditions for testing for fault-coupling. The following theorem provides such a result for a state matrix  $F \in R^{n,n}$ , a fault influence matrix  $G^f \in R^{n,r}$  and an output matrix  $C \in R^{m,n}$ .

**Theorem 5**

If  $rank(C) = m$ ,  $rank(G^f) = r$ ,  $r \leq m$ , and  $(F, G^f, C)$  has no zeros at the origin then the pair  $(F_h, G_2^f, C_h)$  is actuator fault-coupled and detectable with all unstable modes fault coupled if and only if

$$rank \begin{bmatrix} zI_n - F & G^f \\ C & 0_{m \times r} \end{bmatrix} = n + r \tag{27}$$

$$\forall z \in \mathbb{C}, |z| \geq 1$$

**Proof**

Premultiply the matrix in (27) by

$$\begin{bmatrix} U_d^* & 0 \\ 0 & V_c \Sigma_c^{-1} U_{c1}^* \\ 0 & U_{c2}^* \end{bmatrix}$$

and postmultiply by

$$\begin{bmatrix} U_d & 0 \\ 0 & I_r \end{bmatrix}$$

We now have the matrix

$$\begin{bmatrix} zI_q - F_1 & -F_2 & G_1^f \\ -F_3 & zI_{n-q} - F_4 & G_2^f \\ I_q & V_c \Sigma_c^{-1} U_{c1}^* C_2 & 0 \\ 0 & U_{c2}^* C_2 & 0 \end{bmatrix}$$

Next, premultiply by the matrices

$$\begin{bmatrix} I_q & 0 & 0 & 0 \\ 0 & I_{n-q} & F_3 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & I_{m-q} \end{bmatrix}$$

to obtain

$$\begin{bmatrix} zI_q - F_1 & -F_2 & G_1^f \\ 0 & zI_{n-q} - F_h & G_2^f \\ I_q & V_c \Sigma_c^{-1} U_{c1}^* C_2 & 0 \\ 0 & C_h & 0 \end{bmatrix}$$

The matrix is rank deficient only if the matrix

$$\begin{bmatrix} zI_{n-q} - F_h & G_2^f \\ C_h & 0 \end{bmatrix}$$

is rank deficient. The result follows from the PBH tests [12].

◆

**Remark**

- 1) The complex  $z$  values for which the matrix is rank deficient correspond to modes that are not coupled to the fault input. These modes are of no use in fault detection and need not be estimated.
- 2) Theorem 5 can be restated to apply for sensor faults with  $G_2^f$  replaced by the matrix  $-F_3V_c\Sigma_c^{-1}U_{c1}^*E$

If the pair  $(F_h, G_h^f)$  is controllable, then the entire state of the corresponding system is useful for fault detection. If the pair is only stabilizable, then the uncontrollable subsystem is of no use in fault detection and should not be used in state estimation. The controllable subsystem can be obtained by similarity transformation using the controllability matrix of  $(F_h, G_h^f)$ .

**5. Illustrative Examples**

*Example 1*

Consider the following discrete-time double rate system

$$\begin{aligned}
 x(k+1) &= A_d x(k) + B_d^d(k)d(k) + B_d^f f_a(k) \\
 y_i(k) &= C_i x(k) + E_i f_s(k), \quad i = 1,2
 \end{aligned}$$

where,

$$A_d = \begin{bmatrix} -0.45 & 0.45 & 0 & 0 \\ -0.45 & 0 & 0 & 0 \\ 0 & -0.45 & -0.45 & 0 \\ 0.45 & 0 & 0 & -0.45 \end{bmatrix}, \quad B_d^d = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_d^f = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The first set of measurements has two components and is available at the same rate as the input.

$$C_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T_1 = 1$$

The second set of measurements has three components and its period is double the input period.

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad T_2 = 2$$

In the sequel, where two numerical subscripts are used, the first subscript refers to the subsystem number, while the second refers to the component number of a vector.

### 1. The Fast Subsystem (subsystem#1)

For the system with measurements at the same input rate, the disturbance can be decoupled by substituting for  $x_2(k)$  in terms of  $y_1(k)$  and  $x_1(k)$  to obtain the following disturbance-decoupled subsystem as in (15),

$$\bar{x}_{12}^h(k+1) = F_{1h} \bar{x}_{12}^h(k) + G_{1h}^y y_1 + G_{1h}^f \bar{f}_{h1}(k)$$

$$y_{12}(k) = C_{1h} \bar{x}_{12}^h(k)$$

where

$$F_{1h} = \begin{bmatrix} -0.9 & 0 & 0 \\ 0.45 & -0.45 & 0 \\ 0.45 & 0 & -0.45 \end{bmatrix}, \quad G_{1h}^y = \begin{bmatrix} 0.45 \\ -0.45 \\ 0 \end{bmatrix}, \quad G_{1h}^f = \begin{bmatrix} -0.45 & 1 \\ 0.45 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{f}_{h1}(k) = \begin{bmatrix} f_{a1} \\ f_{s1} \end{bmatrix}$$

$$C_{1h} = [1 \quad 2 \quad 0], \quad \bar{x}_{12}^h(k) = [x_1(k) \quad x_3(k) \quad x_4(k)]'$$

The observable/controllable subsystem in this case is given by:

$$z_1(k+1) = A_{z1} z_1(k) + B_1^y y_1 + B_1^f f_{h1}(k)$$

$$y_{12}(k) = C_{z1} z_1(k)$$

where

$$A_{z1} = \begin{bmatrix} -0.9 & 0 \\ 0.45 & -0.45 \end{bmatrix}, \quad B_1^y = \begin{bmatrix} 0.45 \\ -0.45 \end{bmatrix}, \quad B_1^f = \begin{bmatrix} -0.45 & 1 \\ 0.45 & 0 \end{bmatrix}, \quad C_{z1} = [1 \quad 2]$$

$$z_1(k) = [x_1(k) \quad x_3(k)]'$$

The observer for this subsystem (fast rate observer) is constructed as follows,

$$\begin{aligned}\hat{\bar{z}}_1(k+1) &= A_{z_1}\hat{\bar{z}}_1(k) + B_1^y y_1 + L_1[y_{12} - C_{z_1}\hat{\bar{z}}_1(k)] \\ &= [A_{z_1} - L_1 C_{z_1}]\hat{\bar{z}}_1(k) + B_1^y y_1 + L_1 y_2\end{aligned}$$

and its residual is given by

$$r_1(k) = y_{12}(k) - C_{z_1}\hat{\bar{z}}_1(k)$$

The gain of the observer  $L_1$  can be chosen to result in some desired closed-loop poles. For example, for poles at [0.2, -0.2] the gain is given by

$$L_1 = [1.7111 \quad -1.5306]'$$

## 2. The Slow Subsystem (subsystem #2)

The system with slow rate measurements is given by

$$\begin{aligned}\bar{x}_2(k+1) &= A_2\bar{x}_2(k) + G_2^d \bar{d}(k) + G_2^f \bar{f}_a(k) \\ y_2(k) &= C_2\bar{x}_2(k) + E_2 f_{s_2}(k)\end{aligned}$$

$$\text{Where, } A_2 = A^2 = \begin{bmatrix} 0 & -0.2025 & 0 & 0 \\ 0.2025 & -0.2025 & 0 & 0 \\ 0.2025 & 0.2025 & 0.2025 & 0 \\ -0.405 & 0.2025 & 0 & 0.2025 \end{bmatrix}$$

$$G_2^d = [A_2 B_d^d \quad B_d^d] = \begin{bmatrix} 0.45 & 0 \\ 0 & 1 \\ -0.45 & 0 \\ 0 & 0 \end{bmatrix}, \quad G_2^f = [A_2 B_d^f \quad B_d^f] = \begin{bmatrix} 0 & 1 \\ -0.45 & 1 \\ -0.45 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The disturbance-free, observable/controllable subsystem in this case is given by

$$z_2(k+1) = A_{z_2} z_2(k) + B_2^y y_{21}(k) + B_2^f f_{h_2}(k)$$

$$y_{23}(k) = C_{z_2} z_2(k) + f_s(k)$$

where,

$$A_{z_2} = 0.405, \quad B_2^y = [-0.405 \quad 0.2025], \quad B_2^f = [0.405 \quad 1]$$

$$C_{z_2} = 1, \quad f_{h_2}(k) = [f_s(k) \quad f_a(k)]'$$

$$y_2(k) = [y_{21}(k) \quad y_{22}(k)]', \quad z_2(k) = x_{24}(k)$$

where the subscript outside the parentheses denote the order of the element in the vector.

The observer for this subsystem (slow rate observer) is constructed as follows:

$$\hat{z}_2(k+1) = [A_{z_2} - L_2 C_{z_2}] \hat{z}_2(k) + G_2^y y_2(k) + L_2 y_{23}(k)$$

The observer gain  $L_2$  is chosen to result in a prescribed closed-loop pole. For example, for a closed-loop pole at (0.1) the observer gain is

$$L_2 = 0.305$$

The residual is given by

$$r_2(k) = y_{23}(k) - C_{z_2} \hat{z}_2(k)$$

Clearly, the unknown inputs are decoupled from the residuals. Furthermore, because each residual generator is operated at a single sampling rate, fault diagnosis is possible using standard approaches from the literature [9,11,15].

### *Example 2*

In this example, we show how fault isolation is easily achievable using our scheme by treating all but one fault as disturbances as in [11]. We consider a single subsystem with two simultaneous actuator faults, i.e. one fault is considered as an unknown disturbance. We design

two observers with each observer sensitive to a single fault. The residuals generated in this case will be used for fault detection and isolation.

Let the  $i$ th subsystem be given by

$$x(k+1) = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 2 & 5 \end{bmatrix} x(k) + \begin{bmatrix} 1 & -3 \\ -0.5 & 1 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

where,  $f_1$ , and  $f_2$  are two fault modes which to be isolated.

The measurements for this subsystem are assumed to be,

$$y(k) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k)$$

### Observer #1 (Sensitive to $f_1$ ):

To decouple  $f_2$ , we transform the subsystem to the form

$$x(k+1) = \begin{bmatrix} -1 & -2 & -2.846 \\ 0 & 3 & 4.111 \\ 0.6325 & 1.8974 & 5 \end{bmatrix} x(k) + \begin{bmatrix} -1.107 & 3.162 \\ -0.158 & 0 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 0.316 & 0.949 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k)$$

Substituting for  $x_1(k)$  in terms of  $y_1(k)$  and  $x_2(k)$  we obtain the following  $f_1$  coupled subsystem

$$\begin{bmatrix} x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 3 & 4.111 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} y_1(k) + \begin{bmatrix} -0.158 \\ 0.5 \end{bmatrix} f_1$$

$$y_2(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_2(k) \\ x_3(k) \end{bmatrix}$$

The fault coupled observable part is given by

$$x_3(k+1) = 5x_3(k) + 2y_1(k) + 0.5f_1$$

$$y_2(k) = x_3(k)$$

Finally, the observer for the detection and isolation of the first fault ( $f_1$ ) is given by

$$\hat{x}_3(k+1) = (5-l)\hat{x}_3(k) + 2y_1(k) + ly_2(k)$$

and the residual is given by

$$r(k) = y_2(k) - \hat{x}_3(k)$$

where,  $l$  is some properly selected observer gain, which can be determined to result in a desired pole placement.

### Observer #2 (Sensitive to $f_2$ )

To decouple  $f_1$ , we transform the system to the form

$$x(k+1) = \begin{bmatrix} 0.333 & 2.433 & 4.099 \\ 1.391 & 3.742 & 4.3 \\ 1.058 & 1.034 & 2.925 \end{bmatrix} x(k) + \begin{bmatrix} 1.225 & -2.858 \\ 0.0 & -0.317 \\ 0.0 & 1.317 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$y(k) = \begin{bmatrix} -0.408 & 0.908 & 0.092 \\ 0.408 & 0.092 & 0.908 \end{bmatrix} x(k)$$

Substituting for  $x_1(k)$  in terms of  $y_1(k)$ ,  $x_2(k)$  and  $x_3(k)$  we obtain the following  $f_2$ -coupled observable subsystem

$$\begin{bmatrix} x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 6.837 & 4.612 \\ 3.388 & 3.163 \end{bmatrix} \begin{bmatrix} x_2(k) \\ x_3(k) \end{bmatrix} - \begin{bmatrix} 3.41 \\ 2.59 \end{bmatrix} y_1(k) + \begin{bmatrix} -0.317 \\ 1.317 \end{bmatrix} f_2$$

$$y_1(k) + y_2(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_2(k) \\ x_3(k) \end{bmatrix}$$

Finally we obtain the observer for the detection and isolation of the second fault ( $f_2$ )

$$\begin{bmatrix} \hat{x}_2(k+1) \\ \hat{x}_3(k+1) \end{bmatrix} = \begin{bmatrix} 6.837-l_1 & 4.612-l_1 \\ 3.388-l_2 & 3.163-l_2 \end{bmatrix} \begin{bmatrix} \hat{x}_2(k) \\ \hat{x}_3(k) \end{bmatrix} + \begin{bmatrix} l_1-3.418 & l_1 \\ l_2-2.592 & l_2 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}$$

and the residual

$$r(k) = y_1(k) + y_2(k) - \hat{x}_2(k) - \hat{x}_3(k)$$

where,  $l_1, l_2$  observer gains that can be selected for any desired pole placement.

## 6. Conclusion

This paper presents a fault detection scheme for a class of multirate systems with both sensor and actuator faults. The approach utilizes measurements as they become available to estimate the observable partial state that is sensitive to the faults. This provides maximal and timely utilization of the measurements. The work presented here owes much to the work of Hou and Muller on unknown input observer [11]. Nevertheless, the application to multirate systems and the conditions for fault coupling and fault detection we provide are new. We also include sensor faults, which were not considered in [11]. The paper also introduces the concept of fault-coupled subsystems and provides necessary and sufficient conditions for this property. Although it is not discussed in detail, fault isolation is easily achieved using any standard technique from the literature [15,9,11]. Future efforts will involve extension to other classes of multirate sampled data systems.

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## Figure Caption

Figure 1. Illustration of multirate fault detection and isolation for a two-rate system.